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General multipole expansion of polarization observables in deuteron electrodisintegration

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Abstract. Formal expressions are derived for the multipole expansion of the structure functions of a general polarization observable of exclusive electrodisintegration of the deuteron using a longitudinally polarized beam and/or an oriented target. This allows one to exhibit explicitly the angular dependence of the structure functions by expanding them in terms of the small rotation matrices $d^j_{m'm}(\theta)$, whose coefficients are given in terms of the electromagnetic multipole matrix elements. Furthermore, explicit expressions for the coefficients of the angular distributions of the differential cross-section including multipoles up to $L_{\max} = 3$ are listed in tabular form.

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1 Introduction

The special and fundamental role of the two-nucleon system is well recognized. It plays the same role in nuclear physics as the hydrogen atom in atomic physics and is underlined first of all by the fact that NN-scattering is of crucial importance for fitting realistic NN-potential models. Secondly, the deuteron constitutes the simplest nucleus. It is very weakly bound and allows an exact theoretical treatment, at least in the nonrelativistic regime.

Over the past decade we have made a systematic study of inclusive and exclusive deuteron electrodisintegration with special emphasis on polarization observables [1–6]. The main purpose of this study was to reveal to what extent the use of polarized electrons, polarized targets and polarization analysis of the outgoing nucleons will allow a considerably more thorough and more detailed investigation of the dynamical features of the two-nucleon system than is possible without the use of polarization degrees of freedom (d.o.f.).

With the present work we continue the study of the formal aspects of this reaction presented in [3,6]. In [3] we have formally derived all possible polarization structure functions as an extension to previous work in photodisintegration [7,8]. In view of the large number of observables, we have addressed the question of independent observables in a more general sense in [5] considering a two-body reaction of the type $a+b \to c+d$, for which we have derived a general criterion for the selection of a complete set of independent observables. Subsequently it has been applied in [6] to the electromagnetic deuteron break-up reaction which can be considered as a two-body reaction in the one-photon-exchange approximation.

It is the aim of the present work to derive the multipole expansion of the observables of this reaction, which allows one to represent any observable as an expansion in terms of the small rotation matrices $d^j_{m'm}(\theta)$, whose coefficients are determined uniquely by the electromagnetic transition multipole matrix elements between the deuteron ground state and the various partial waves of the outgoing two-nucleon scattering state. Our approach is based on earlier work in deuteron photodisintegration [7] in which the multipole expansions of the unpolarized differential cross-section and of the outgoing-nucleon polarization without target orientation of [9,10] have been generalized to all possible polarization observables. Analogous techniques have been applied in [11] for the description of polarization effects in (γ, N)-reactions on nuclei and in [12] for polarization observables in coincidence electron scattering from nuclei. In [11] only photon polarization degrees of freedom and outgoing-nucleon polarization is considered without including effects from target polarization, whereas in [12] the latter are treated, too. In particular, in [12] detailed

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expressions are given for the differential cross-section of deuteron electrodisintegration including beam and target polarization and for one-nucleon recoil polarization without target orientation.

A multipole decomposition will be very useful for a detailed comparison between theory and experiment. Past experience in deuteron photodisintegration has shown that a study of the multipole decomposition of angular distributions often helps ascertain the reasons for any serious discrepancy between theory and experiment. However, one should keep in mind that such an analysis is managable only if the multipole expansion converges rapidly so that not too many multipoles contribute significantly. This is certainly true for photodisintegration at low and medium energies, say up to the Δ-resonance region, but not for electrodisintegration in general, because for energies and momentum transfers along the quasifree ridge, the multipole expansion converges slowly. But this is the region where the influence of final-state interactions (fsi) is minimal and thus this is not the best region for testing the NN-interaction. Away from the quasifree ridge the multipole series converges quite rapidly, at least below the ridge, for example at a final-state c.m. energy of 120 MeV and $\vec{q}^{\,2} < 2$ fm^2, as has been shown in [13]. On the other hand, this is just the interesting region where fsi and two-body currents become significant allowing a much more stringent test of a NN-potential model and its associated two-body current operator. Thus, a multipole analysis can become an important tool for a detailed analysis.

First, we will briefly review in the next section the general structure of an observable and its representation in terms of a bilinear Hermitean form in the current matrix elements. Starting from the multipole expansion of the current, we then derive in sect. 3 formal expressions for the coefficients of the expansion of an observable in terms of the small rotation matrices $d^j_{m'm}(\theta)$. Some explicit expressions are collected in two appendices.

2 General form of an observable

We will begin with a brief review of the general formalism for an observable in $e + d \to e' + n + p$ as derived in detail in [3]. A different approach has been used in [14] but there is a one-to-one correspondence between the observables of [14] and ours as shown in detail in the appendix A of [6]. In the one-photon-exchange approximation, the most general form of an observable "X" in $d(e, e'N)N$ and $d(e, e'np)$ is given by

$$\mathcal{O}(\Omega_X) = 3\, c(k_1^{\rm lab}, k_2^{\rm lab})\, {\rm tr}(T^\dagger \Omega_X T \rho_i)\,, \tag{1}$$

where

$$c(k_1^{\rm lab}, k_2^{\rm lab}) = \frac{\alpha}{6\pi^2} \frac{k_2^{\rm lab}}{k_1^{\rm lab} q_\nu^4}\,, \tag{2}$$

with α denoting the fine structure constant and q_ν^2 the four-momentum transfer squared ($q = k_1 - k_2$). Here, $\vec{k}_1$ and $\vec{k}_2$ denote the momenta of incoming and scattered electron, respectively.

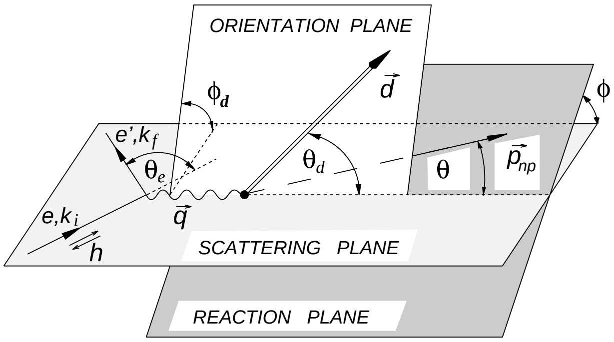

Fig. 1. Geometry of exclusive electron-deuteron scattering with polarized electrons and an oriented deuteron target. The relative np-momentum, denoted by $\vec{p}_{np}$, is characterized by the angles $\theta = \theta_{np}$ and $\phi = \phi_{np}$, where the deuteron orientation axis, denoted by $\vec{d}$, is specified by the angles θ_d and ϕ_d.

The scattering geometry is illustrated in fig. 1, in which we distinguish three different planes which all intersect in one line as defined by the momentum transfer $\vec{q}$, namely, the scattering plane, the reaction plane, and the orientation plane containing the axis of orientation of a polarized deuteron. The principal frames of reference are associated with the scattering plane, namely, the laboratory frame and the c.m. frame of the final two nucleons, which is related to the former one by a boost along $\vec{q}$. The z-axis is chosen along $\vec{q}$ and the y-axis in the direction of $\vec{k}_1 \times \vec{k}_2$ and hence perpendicular to the scattering plane, and the x-axis such as to form a right-handed system. With respect to the c.m. frame, we will denote throughout this paper by θ and ϕ the spherical angles of the relative momentum $\vec{p}_{np} = (p^{\rm c.m.}, \theta, \phi)$. Thus, the spherical angles of proton and neutron momenta in this frame are $\theta_p^{\rm c.m.} = \theta$, $\phi_p^{\rm c.m.} = \phi$ and $\theta_n^{\rm c.m.} = \pi - \theta$, $\phi_n^{\rm c.m.} = \phi + \pi$ (see fig. 1). The final hadronic state is furthermore characterized by the excitation energy ε_{np}. Finally, θ_d and ϕ_d denote the spherical angles of the deuteron orientation axis.

The T-matrix in (1) is related to the current matrix element between the initial deuteron state and the final np-scattering state. Characterizing the initial deuteron state by its spin projection m_d on $\vec{q}$ and taking as spin degrees of the final state the total spin s and m_s its projection on the relative np-momentum $\vec{p}_{np}$ in the final np-c.m. system, one obtains for the T-matrix between the initial deuteron state $|m_d\rangle$ and the final np-scattering state $|sm_s\rangle$

$$T_{sm_s\lambda m_d}(\theta,\phi) = -\pi\sqrt{2\alpha\, p_{np} E^{\rm c.m.} E_d^{\rm c.m.}/M_d} \times \langle sm_s|\hat{J}_\lambda(\vec{q})|m_d\rangle = \tag{3a}$$

$$e^{i(\lambda+m_d)\phi}\, t_{sm_s\lambda m_d}(\theta)\,, \tag{3b}$$

where $E^{\rm c.m.} = \sqrt{M^2 + p_{np}^2}$ and $E_d^{\rm c.m.} = \sqrt{M_d^2 + q^2}$ denote the nucleon and deuteron c.m. energies, respectively. We would like to remark that the choice of the coupled-spin representation of the T-matrix is not essential. One could as well take the uncoupled-spin representation $T_{\lambda_p\lambda_n\lambda m_d}$, where $\lambda_{p/n}$ denote the spin projections of the proton and the neutron on the relative momentum, respectively. It is related to the coupled-spin representation by a Clebsch-Gordan coefficient

$$T_{\lambda_p \lambda_n \lambda m_d} = \sum_{sm_s} \left(\frac{1}{2}\lambda_p \frac{1}{2}\lambda_n | sm_s \right) T_{sm_s \lambda m_d} \,. \quad (4)$$

The structure functions describing any observable do not depend on the choice of representation, only their formal appearance in terms of T-matrix elements will be different in the different representations.

Each observable X is represented by a pair $X = (\alpha' \, \alpha)$ with α', $\alpha = 0, \ldots, 3$ referring either to no polarization analysis of the outgoing nucleons (α', $\alpha = 0$) or to their polarization components (α', $\alpha = 1, 2, 3$), and Ω_X is an associated operator in the spin space of each of the two nucleons. In detail, if no polarization analysis of the outgoing nucleons is performed, one has

$$\Omega_1 = \Omega_{00} = \mathbb{1}_2(p) \otimes \mathbb{1}_2(n) \,, \quad (5)$$

and if the polarization component x_i of the proton or the neutron, respectively, is measured,

$$\Omega_{i\,0} = \sigma_{x_i}(p) \otimes \mathbb{1}_2(n) \quad \text{or}$$
$$\Omega_{0\,i} = \mathbb{1}_2(p) \otimes \sigma_{x_i}(n) \,, \quad (i = 1, 2, 3) \,. \quad (6)$$

Finally, the polarization components $x_i(p)$ and $x_j(n)$ of both particles are represented by

$$\Omega_{i\,j} = \sigma_{x_i}(p) \otimes \sigma_{x_j}(n) \,. \quad (7)$$

The resulting observables are listed in table 1 and are divided into two sets, called A and B, according to their behaviour under a parity transformation [8].

Since the T-matrix of this reaction is calculated in the np-c.m. system, the spin operators refer to the same reference frame. In the Madison convention the polarization components of the outgoing particles refer to a frame of reference, for which the z-axis is taken along the particle momentum, *i.e.*, in the reaction plane, the y-axis along $\vec{q} \times \vec{p}_i$, *i.e.*, perpendicular to the reaction plane, and the x-axis is then determined by the requirement to form a right-handed system. However, one should keep in mind that the spin operators of both particles refer to the same coordinate system with z-axis parallel to $\vec{p}_{np}$ and y-axis along $\vec{q} \times \vec{p}_{np}$, *i.e.*, perpendicular to the reaction plane. Thus, the polarization components of the proton are chosen according to the Madison convention while for the neutron the y- and z-components of its polarization have to be reversed in order to comply with this convention.

Furthermore, in order to account for a possible target orientation, the initial-state density matrix in (1) comprises besides the density matrix of the exchanged virtual photon the deuteron density matrix ρ^d, which we take in the form

$$\rho^d_{m_d\, m_d'} = \frac{1}{\sqrt{3}} (-)^{1-m_d} \sum_{I\,M} \hat{I} \begin{pmatrix} 1 & 1 & I \\ m_d' & -m_d & M \end{pmatrix}$$
$$\times P^d_I e^{-iM\phi_d} d^I_{M0}(\theta_d) \,, \quad (8)$$

where $P^d_0 = 1$. We use throughout the notation $\hat{I} = \sqrt{2I+1}$. In (8) we have assumed that the deuteron density matrix is diagonal with respect to an axis $\hat{d}$ which is called the orientation axis. Therefore, the deuteron target is characterized by four parameters, namely the vector and tensor polarizations P^d_1 and P^d_2, respectively, and by the orientation angles θ_d and ϕ_d describing the direction of the orientation axis $\hat{d}$ of the polarized deuteron target with respect to the coordinate system associated with the scattering plane (see fig. 1). Note that the deuteron density matrix undergoes no change in the transformation from the laboratory to the c.m. system, since the boost to the c.m. system is collinear with the deuteron quantization axis [15].

Any observable X in $d(e, e'N)N$ and $d(e, e'np)$ can be represented in terms of structure functions $f^{(\prime)IM\pm}_a(X)$ ($a \in \{L, T, LT, TT\}$) and is given by

$$\mathcal{O}(\Omega_X) = c(k_1^{\text{lab}},\, k_2^{\text{lab}}) \sum_{I=0}^{2} P^d_I \sum_{M=0}^{I} \Big\{ \Big(\rho_L f^{IM}_L(X)$$
$$+ \rho_T f^{IM}_T(X) + \rho_{LT} f^{IM+}_{LT}(X) \cos\phi$$
$$+ \rho_{TT} f^{IM+}_{TT}(X) \cos 2\phi \Big) \cos\Big(M\tilde{\phi} - \bar{\delta}^X_I \frac{\pi}{2} \Big)$$
$$- \Big(\rho_{LT} f^{IM-}_{LT}(X) \sin\phi + \rho_{TT} f^{IM-}_{TT}(X) \sin 2\phi \Big)$$
$$\times \sin\Big(M\tilde{\phi} - \bar{\delta}^X_I \frac{\pi}{2} \Big)$$
$$+ h \Big[\Big(\rho'_T f'^{IM}_T(X) + \rho'_{LT} f'^{IM-}_{LT}(X) \cos\phi \Big)$$
$$\times \sin\Big(M\tilde{\phi} - \bar{\delta}^X_I \frac{\pi}{2} \Big)$$
$$+ \rho'_{LT} f'^{IM+}_{LT}(X) \sin\phi \cos\Big(M\tilde{\phi} - \bar{\delta}^X_I \frac{\pi}{2} \Big) \Big] \Big\} d^I_{M0}(\theta_d) \,, \quad (9)$$

where $d^j_{m'm}(\theta)$ denotes the small d-function of the rotation matrices [16], and $\tilde{\phi} = \phi - \phi_d$.

In particular, one obtains for $X = (00)$ the unpolarized cross-section as

$$S_0 = c(k_1^{\text{lab}},\, k_2^{\text{lab}})$$
$$\times (\rho_L f_L + \rho_T f_T + \rho_{LT} f_{LT} \cos\phi + \rho_{TT} f_{TT} \cos 2\phi) \,, \quad (10)$$

using as a shorthand $f_a = f^{00+}_a(1)$. One should remember that the nucleon angles and polarization components refer to the c.m. frame.

The kinematic factors of the virtual photon density matrix ρ_a and ρ'_a are given by the well-known expressions (note $Q^2 = -q^2_\nu > 0$)

$$\rho_L = \beta^2 Q^2 \tfrac{\xi^2}{2\zeta} \,, \qquad \rho_T = \tfrac{1}{2} Q^2 \Big(1 + \tfrac{\xi}{2\zeta} \Big) \,,$$
$$\rho_{LT} = \beta Q^2 \tfrac{\xi}{\zeta} \sqrt{\tfrac{\zeta+\xi}{8}} \,, \qquad \rho_{TT} = -Q^2 \tfrac{\xi}{4\zeta} \,, \quad (11)$$
$$\rho'_{LT} = \tfrac{1}{2} \beta \tfrac{Q^2}{\sqrt{2\zeta}} \xi \,, \qquad \rho'_T = \tfrac{1}{2} Q^2 \sqrt{\tfrac{\zeta+\xi}{\zeta}} \,,$$

Table 1. Notation for the Cartesian components of the spin observables and their division into sets A and B.

Observable	1	x_p	y_p	z_p	x_n	y_n	z_n		
Set	A	B	A	B	B	A	B		
Observable	x_px_n	x_py_n	x_pz_n	y_px_n	y_py_n	y_pz_n	z_px_n	z_py_n	z_pz_n
Set	A	B	A	B	A	B	A	B	A

with

$$\beta = \frac{|\vec{q}^{\,\mathrm{lab}}|}{|\vec{q}^{\,c}|}, \quad \xi = \frac{Q^2}{(\vec{q}^{\,\mathrm{lab}})^2}, \quad \zeta = \tan^2\left(\frac{\theta_e^{\mathrm{lab}}}{2}\right), \tag{12}$$

where β expresses the boost from the laboratory system to the frame in which the hadronic current is evaluated and $\vec{q}^{\,c}$ denotes the momentum transfer in this frame.

The explicit form of the structure functions in (9) has been derived in [3] and is given by

$$f_L^{IM}(X) = \frac{2}{1+\delta_{M,0}} \Re e\left(i^{\bar\delta_I^X} \mathcal{U}_X^{00IM}\right), \tag{13a}$$

$$f_T^{IM}(X) = \frac{4}{1+\delta_{M,0}} \Re e\left(i^{\bar\delta_I^X} \mathcal{U}_X^{11IM}\right), \tag{13b}$$

$$f_{LT}^{IM\pm}(X) = \frac{4}{1+\delta_{M,0}} \times \Re e\left[i^{\bar\delta_I^X}\left(\mathcal{U}_X^{01IM} \pm (-)^{I+M+\delta_{X,\mathrm{B}}} \mathcal{U}_X^{01I-M}\right)\right], \tag{13c}$$

$$f_{TT}^{IM\pm}(X) = \frac{2}{1+\delta_{M,0}} \times \Re e\left[i^{\bar\delta_I^X}\left(\mathcal{U}_X^{-11IM} \pm (-)^{I+M+\delta_{X,\mathrm{B}}} \mathcal{U}_X^{-11I-M}\right)\right], \tag{13d}$$

$$f_T'^{IM}(X) = \frac{4}{1+\delta_{M,0}} \Re e\left(i^{1+\bar\delta_I^X} \mathcal{U}_X^{11IM}\right), \tag{13e}$$

$$f_{LT}'^{IM\pm}(X) = \frac{4}{1+\delta_{M,0}} \times \Re e\left[i^{1+\bar\delta_I^X}\left(\mathcal{U}_X^{01IM} \pm (-)^{I+M+\delta_{X,\mathrm{B}}} \mathcal{U}_X^{01I-M}\right)\right]. \tag{13f}$$

Here $\bar\delta_I^X$ is defined by

$$\bar\delta_I^X := (\delta_{X,\mathrm{B}} - \delta_{I,1})^2, \text{ with } \delta_{X,\mathrm{B}} := \begin{Bmatrix} 1, \text{ for } X \in \mathrm{B} \\ 0, \text{ for } X \in \mathrm{A} \end{Bmatrix}, \tag{14}$$

distinguishing the two sets of observables A and B. In the foregoing expressions, the $\mathcal{U}$'s are given as bilinear Hermitean forms in the reaction matrix elements, *i.e.*, for $X = (\alpha'\alpha)$

$$\mathcal{U}_{\alpha'\alpha}^{\lambda'\lambda IM} = \sum_{s'm_s'm_d'sm_sm_d} t^*_{s'm_s'\lambda'm_d'} \times \langle s'm_s'|\sigma_{\alpha'}(p)\sigma_\alpha(n)|sm_s\rangle t_{sm_s\lambda m_d} \left\langle m_d \middle| \tau_M^{[I]} \middle| m_d' \right\rangle, \tag{15}$$

where the irreducible spin operators $\tau^{[I]}$ with respect to the deuteron spin space are defined by their irreducible matrix elements

$$\langle 1 | \tau^{[I]} | 1 \rangle = \sqrt{3}\,\hat{I}, \quad (I = 0,1,2). \tag{16}$$

Explicitly one has for the $\mathcal{U}$'s

$$\mathcal{U}_{\alpha'\alpha}^{\lambda'\lambda IM} = 2 \sum_{\tau'\nu'\tau\nu} (-)^{\tau'+\tau} \hat\tau' \hat\tau\, s_{\alpha'}^{\tau'\nu'} \times s_\alpha^{\tau\nu} \sum_{S\sigma} (-)^\sigma \hat{S}^2 \begin{pmatrix} \tau' & \tau & S \\ \nu' & \nu & -\sigma \end{pmatrix} \times \sum_{s's} \hat{s}'\hat{s} \begin{Bmatrix} \frac12 & \frac12 & \tau' \\ \frac12 & \frac12 & \tau \\ s' & s & S \end{Bmatrix} u_{\lambda'\lambda IM}^{s'sS\sigma}, \tag{17}$$

with

$$u_{\lambda'\lambda IM}^{s'sS\sigma} = \hat{I}\sqrt{3} \sum_{m_s'm_sm'm} (-)^{1-m+s'-m_s'} \begin{pmatrix} 1 & 1 & I \\ m' & -m & M \end{pmatrix} \times \begin{pmatrix} s' & s & S \\ m_s' & -m_s & -\sigma \end{pmatrix} t^*_{s'm_s'\lambda'm'} t_{sm_s\lambda m}, \tag{18}$$

and $s_\alpha^{\tau\nu}$ transforms the spherical components of the spin operators $\sigma_\nu^{[\tau]}$ ($\tau = 0,1$) to Cartesian ones σ_α. It is given by

$$s_\alpha^{\tau\nu} = \bar{c}(\alpha)\,\delta_{\tau,\tilde\tau(\alpha)}\left(\delta_{\nu,\tilde\nu(\alpha)} + \hat{c}(\alpha)\,\delta_{\nu,-\tilde\nu(\alpha)}\right), \tag{19}$$

with

$$\hat{c}(\alpha) = \delta_{\alpha,2} - \delta_{\alpha,1}, \quad \bar{c}(\alpha) = \begin{Bmatrix} 1, & \text{for } \alpha = 0,3 \\ \frac{i^{-\alpha-1}}{\sqrt2}, & \text{for } \alpha = 1,2 \end{Bmatrix},$$

$$\tilde\tau(\alpha) = 1 - \delta_{\alpha,0}, \qquad \tilde\nu(\alpha) = \begin{Bmatrix} 0, \text{ for } \alpha = 0,3 \\ 1, \text{ for } \alpha = 1,2 \end{Bmatrix}. \tag{20}$$

For later purpose we note the following properties:

$$(s_\alpha^{\tau\nu})^* = (-)^{\delta_{\alpha,2}} s_\alpha^{\tau\nu}, \qquad s_\alpha^{\tau-\nu} = (-)^{\delta_{\alpha,1}} s_\alpha^{\tau\nu}, \tag{21}$$

and

$$(-)^\tau s_\alpha^{\tau\nu} = (-)^{1-\delta_{\alpha,0}} s_\alpha^{\tau\nu}, \qquad (-)^\nu s_\alpha^{\tau\nu} = (-)^{\delta_{\alpha,1}+\delta_{\alpha,2}} s_\alpha^{\tau\nu}. \tag{22}$$

The $\mathcal{U}$ transforms under complex conjugation as

$$\left(\mathcal{U}_{\alpha'\alpha}^{\lambda'\lambda IM}\right)^* = (-)^M \mathcal{U}_{\alpha'\alpha}^{\lambda\lambda' I-M}. \tag{23}$$

Note that $f_a^{00-}(X)$, $f_a^{20,-}(X)$ and $f_a^{10+}(X)$ vanish identically. For this reason we use the notation $f_a(X)$, $f_a^{10}(X)$ and $f_a^{20}(X)$ instead of $f_a^{00+}(X)$, $f_a^{10-}(X)$ and $f_a^{20+}(X)$, respectively.

Table 2. Listing of the matrix $U^j_{ls\mu}$.

l	s	$\mu = 1$	2	3	4
$j-1$	1	$\cos\epsilon_j$	0	$-\sin\epsilon_j$	0
j	0	0	1	0	0
$j+1$	1	$\sin\epsilon_j$	0	$\cos\epsilon_j$	0
j	1	0	0	0	1

The structure functions $f_a^{(\prime)\,IM(\pm)}(X)$ contain the complete information on the dynamical properties of the NN system available in deuteron electrodisintegration. They are functions of θ, the relative np-energy ε_{np} and the three-momentum transfer squared $(\vec{q}^{\,\mathrm{c.m.}})^2$. Both the np-energy ε_{np} and $(\vec{q}^{\,\mathrm{c.m.}})^2$ are taken in the c.m. system.

Finally, we would like to remark that for real photons only the transverse-structure functions contribute. The corresponding photoabsorption cross-section is obtained from (1) by the replacements $c(k_1^{\mathrm{lab}}, k_2^{\mathrm{lab}}) \to 1/3$ and

$$\begin{aligned} &\rho_L \to 0, \quad \rho_{LT} \to 0, \qquad \rho'_{LT} \to 0, \\ &\rho_T \to \tfrac{1}{2}, \quad h\rho'_T \to \tfrac{1}{2} P_c^\gamma, \quad \rho_{TT} \to -\tfrac{1}{2} P_1^\gamma\,, \end{aligned} \tag{24}$$

where P_1^γ and P_c^γ denote the degree of linear and circular photon polarization, respectively.

3 Multipole expansion

In order to have a convenient parametrization of the angular behaviour of the structure functions it is useful to expand them in terms of the small rotation matrices $d^j_{m'm}(\theta)$. This will also facilitate the analysis of the contributions of the various electric and magnetic transition multipole moments to the different structure functions. It is achieved with the help of the multipole expansion for the t-matrix. We take the outgoing np-state in the form of the Blatt-Biedenharn convention [17]

$$\begin{aligned} |\vec{p}\, s\, m_s\rangle^{(-)} = \sum_{\mu j m_j l} &\hat{l}\,(l0sm_s|jm_s) \\ &\times e^{-i\delta^j_\mu}\, U^j_{ls\mu}\, D^j_{m_j m_s}(R)\,|\mu j m_j\rangle\,, \end{aligned} \tag{25}$$

where δ^j_μ denotes the hadronic-phase shift, and the matrix $U^j_{ls\mu}$ is determined by the mixing parameters ϵ_j as listed in table 2. Furthermore, R rotates the chosen quantization axis into the direction of the relative np-momentum $\vec{p}$. Here, the partial waves

$$|\mu j m_j\rangle = \sum_{l's'} U^j_{l's'\mu}\,|\mu(l's')jm_j\rangle \tag{26}$$

are the solutions of a system of coupled equations of NN-scattering. Strictly speaking, such a representation is valid only for energies below the pion production threshold, because above this threshold the np-channel is coupled to the $NN\pi$-channel. However, if one is not interested in the pionic channels, one can project them out at the price that the phase shifts become complex, where the imaginary parts describe the inelasticities. A further consequence is that the radial functions, which were real below the pion threshold, become complex too.

Although an uncoupled representation like the helicity basis [18] is preferred in high-energy reactions, we have purposely chosen the coupled representation because NN-scattering data like phase shifts and mixing parameters, to which all modern high-precision NN-potentials are fitted, are based on it. These potentials are constructed in order to describe NN-scattering data at low and medium energies to a high degree of accuracy and thus it is quite natural to make this choice for the multipole analysis in order to provide a more stringent comparison between theory and experiment and thus a finer test of these high-precision potentials. In addition, we would like to remark that even though the coupled-spin representation originally was introduced for a nonrelativistic description, it still can be maintained in the case that leading-order relativistic contributions are included. Furthermore, we will briefly show in appendix A, where we give the multipole expansion for an uncoupled representation (valid also for a fully covariant description), that one can still introduce formally a (ls)-representation.

In the convention (25), the multipole expansion of the t-matrix reads

$$\begin{aligned} t_{sm_s\lambda m_d}(\theta) = &(-)^\lambda \sqrt{1+\delta_{\lambda,0}} \sum_{Ljm_j\mu} \frac{\hat{l}}{\hat{j}}(1m_dL\lambda|jm_j) \\ &\times (l0sm_s|jm_s)\mathcal{O}^{L\lambda}(\mu jls) d^j_{m_j m_s}(\theta) = \\ &(-)^{1+m_d+m_s}\sqrt{1+\delta_{\lambda,0}} \sum_{Ljm_j\mu} (-)^{L+l+s}\hat{l}\,\hat{j} \\ &\times \begin{pmatrix} 1 & L & j \\ m_d & \lambda & -m_j \end{pmatrix} \begin{pmatrix} l & s & j \\ 0 & m_s & -m_s \end{pmatrix} \mathcal{O}^{L\lambda}(\mu jls)\, d^j_{m_j m_s}(\theta)\,, \end{aligned} \tag{27}$$

with

$$\mathcal{O}^{L\lambda}(\mu jls) = \sqrt{4\pi}\, e^{i\delta^j_\mu} U^j_{ls,\mu} N^L_\lambda(\mu j)\,, \tag{28}$$

and

$$N^L_\lambda(\mu j) = \delta_{|\lambda|,1}\Big(E^L(\mu j) + \lambda M^L(\mu j)\Big) + \delta_{\lambda,0} C^L(\mu j)\,, \tag{29}$$

where $E^L(\mu j)$, $M^L(\mu j)$ and $C^L(\mu j)$ denote the reduced electric, magnetic and charge multipole matrix elements, respectively, between the deuteron state and a final-state partial wave $|\mu j\rangle$ in the Blatt-Biedenharn parametrization. If the time reversal invariance holds, these matrix elements can be made real by a proper phase convention for energies below the pion production threshold. This is not possible above this threshold. Parity conservation implies the selection rules

$$(C/E)^L(\mu j) = 0\,, \qquad \text{for} \quad (-)^{L+j+\mu} = -1\,, \tag{30a}$$

$$M^L(\mu j) = 0\,, \qquad \text{for} \quad (-)^{L+j+\mu} = 1\,, \tag{30b}$$

which with $(-)^{\mu+j+l} = 1$ leads to the relation

$$\mathcal{O}^{L-\lambda}(\mu jls) = (-)^{L+l}\mathcal{O}^{L\lambda}(\mu jls)\,. \tag{31}$$

In order to obtain the multipole expansion of the quantities $\mathcal{U}^{\lambda'\lambda IM}_{\alpha'\alpha}$ in (17), we generalize the approach in photodisintegration [7] to include also the charge contributions. Thus, we will start with the multipole expansion of $u^{s'sS\sigma}_{\lambda'\lambda IM}(\theta)$. Inserting the multipole expansion of the t-matrix of (27) into (18) one first obtains a rather complicated expression

$$\begin{aligned} u^{s'sS\sigma}_{\lambda'\lambda IM} &= \hat{I}\sqrt{3(1+\delta_{\lambda',0})(1+\delta_{\lambda,0})} \\ &\times \sum_{m'_s m_s m' m} (-)^{1-m+s'-m'_s} \begin{pmatrix} 1 & 1 & I \\ m' & -m & M \end{pmatrix} \begin{pmatrix} s' & s & S \\ m'_s & -m_s & -\sigma \end{pmatrix} \\ &\times \sum_{L'l'j'm'_j\mu'} (-)^{L'+l'+s'+m'_s+m'} \hat{l}'\,\hat{j}' \begin{pmatrix} 1 & L' & j' \\ m' & \lambda' & -m'_j \end{pmatrix} \\ &\times \begin{pmatrix} l' & s' & j' \\ 0 & m'_s & -m'_s \end{pmatrix} \mathcal{O}^{L'\lambda'}(\mu' j' l' s')^*\, d^{j'}_{m'_j m'_s}(\theta) \\ &\times \sum_{Lljm_j\mu} (-)^{L+l+s+m_s+m} \hat{l}\,\hat{j} \begin{pmatrix} 1 & L & j \\ m & \lambda & -m_j \end{pmatrix} \\ &\times \begin{pmatrix} l & s & j \\ 0 & m_s & -m_s \end{pmatrix} \mathcal{O}^{L\lambda}(\mu j l s)\, d^{j}_{m_j m_s}(\theta)\,, \end{aligned} \tag{32}$$

which, however, can be simplified considerably with the help of the Clebsch-Gordan series of the $d^j_{m'm}$-functions

$$\begin{aligned} d^{j'}_{m'_j m'_s}(\theta)\, d^{j}_{m_j m_s}(\theta) &= \\ (-)^{m_s-m_j} \sum_K \hat{K}^2 & \begin{pmatrix} j' & j & K \\ m'_s & -m_s & m_s-m'_s \end{pmatrix} \\ \times & \begin{pmatrix} j' & j & K \\ m'_j & -m_j & m_j-m'_j \end{pmatrix} d^K_{m_j-m'_j,\, m_s-m'_s}(\theta)\,, \end{aligned} \tag{33}$$

and a sum rule for $3j$-symbols yielding

$$\begin{aligned} \sum_{m'_s m_s} & \begin{pmatrix} s' & s & S \\ m'_s & -m_s & -\sigma \end{pmatrix} \begin{pmatrix} l & s & j \\ 0 & m_s & -m_s \end{pmatrix} \\ & \times \begin{pmatrix} l' & s' & j' \\ 0 & m'_s & -m'_s \end{pmatrix} \begin{pmatrix} j' & j & K \\ m'_s & -m_s & -\sigma \end{pmatrix} = \\ & (-)^{l'+s'+j+K} \sum_{K'} \hat{K}'^2 \begin{pmatrix} S & K & K' \\ \sigma & -\sigma & 0 \end{pmatrix} \\ & \times \begin{pmatrix} K' & l & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} S & K & K' \\ s & j & l \\ s' & j' & l' \end{Bmatrix}, \end{aligned} \tag{34}$$

$$\begin{aligned} \sum_{m'_j m_j m' m} & (-)^{m+m'_j} \begin{pmatrix} 1 & 1 & I \\ m' & -m & M \end{pmatrix} \begin{pmatrix} 1 & L & j \\ m & \lambda & -m_j \end{pmatrix} \\ & \times \begin{pmatrix} 1 & L' & j' \\ m' & \lambda' & -m'_j \end{pmatrix} \begin{pmatrix} j' & j & K \\ m'_j & -m_j & \kappa \end{pmatrix} = \\ & (-)^{1+L'+j+K+M+\lambda'} \sum_J \hat{J}^2 \begin{pmatrix} J & I & K \\ \lambda-\lambda' & M & -\kappa \end{pmatrix} \\ & \times \begin{pmatrix} L' & L & J \\ \lambda' & -\lambda & \lambda-\lambda' \end{pmatrix} \begin{Bmatrix} j' & j & K \\ L' & L & J \\ 1 & 1 & I \end{Bmatrix}, \end{aligned} \tag{35}$$

where $\kappa = \lambda - \lambda' + M$. Then (32) can be written in the form

$$u^{s'sS\sigma}_{\lambda'\lambda IM}(\theta) = \sum_K u^{s'sS\sigma,\,K}_{\lambda'\lambda IM}\, d^K_{\lambda'-\lambda-M,\sigma}(\theta)\,, \tag{36}$$

where the coefficients are given in terms of the e.m. multipole moments

$$\begin{aligned} u^{s'sS\sigma,\,K}_{\lambda'\lambda IM} &= \frac{1}{2}(-)^{s'+s+\sigma} \sum_{L'j'Lj} \mathcal{C}^{\lambda'\lambda IM,\,K}(L'j'Lj) \\ &\times \sum_{K'} \hat{K}'^2 \begin{pmatrix} S & K & K' \\ \sigma & -\sigma & 0 \end{pmatrix} \sum_{\mu'l'\mu l} (-)^l\, \hat{l}\,\hat{l}' \begin{pmatrix} K' & l & l' \\ 0 & 0 & 0 \end{pmatrix} \\ &\times \begin{Bmatrix} S & K & K' \\ s & j & l \\ s' & j' & l' \end{Bmatrix} \mathcal{O}^{L'\lambda'}(\mu' j' l' s')^* \mathcal{O}^{L\lambda}(\mu j l s), \end{aligned} \tag{37}$$

with

$$\begin{aligned} \mathcal{C}^{\lambda'\lambda IM,\,K}&(L'j'Lj) = \\ & (-)^{\lambda'+L}\, 2\sqrt{3\,(1+\delta_{\lambda',0})(1+\delta_{\lambda,0})}\,\hat{j}'\,\hat{j}\,\hat{I}\,\hat{K}^2 \\ & \times \sum_J \hat{J}^2 \begin{pmatrix} J & I & K \\ \lambda-\lambda' & M & \lambda'-\lambda-M \end{pmatrix} \\ & \times \begin{pmatrix} L' & L & J \\ \lambda' & -\lambda & \lambda-\lambda' \end{pmatrix} \begin{Bmatrix} j' & j & K \\ L' & L & J \\ 1 & 1 & I \end{Bmatrix}. \end{aligned} \tag{38}$$

The latter coefficients possess the symmetry properties

$$\begin{aligned} \mathcal{C}^{-\lambda'-\lambda IM,\,K}(L'j'Lj) &= \\ & (-)^{L'+L+I+K}\, \mathcal{C}^{\lambda'\lambda I-M,\,K}(L'j'Lj)\,, \end{aligned} \tag{39a}$$

$$\begin{aligned} \mathcal{C}^{\lambda'\lambda IM,\,K}(LjL'j') &= \\ & (-)^{\lambda'+\lambda+j'+j}\, \mathcal{C}^{\lambda\lambda' I-M,\,K}(L'j'Lj)\,. \end{aligned} \tag{39b}$$

The coefficients (37) vanish obviously for $K < |\lambda'-\lambda-M|$. Furthermore, for a given K, according to the $9j$-symbol in (38), only those multipoles L' and L contribute to (37) which fulfil the conditions $|L'-L| \le K+I$ and $L'+L \ge |K-I|$ simultaneously. On the other hand, limiting the multipoles to $L', L \le L_{\max}$, the coefficients vanish for $K > 2L_{\max}+I$.

Furthermore, one finds easily for the coefficients $u^{s'sS\sigma,\,K}_{\lambda'\lambda IM}$ the symmetry properties

$$u^{s'sS\sigma,\,K}_{-\lambda'-\lambda IM} = (-)^{I+S}\, u^{s'sS-\sigma,\,K}_{\lambda'\lambda I-M} \tag{40a}$$

$$\begin{aligned} (u^{s'sS\sigma,\,K}_{\lambda'\lambda IM})^* &= (-)^{s'+s+\lambda'-\lambda}\, u^{ss'S-\sigma,\,K}_{\lambda\lambda' I-M} = \\ & (-)^{I+S+s'+s+\lambda'-\lambda}\, u^{ss'S\sigma,\,K}_{-\lambda-\lambda' IM}\,, \end{aligned} \tag{40b}$$

which follow directly from (31), (37) and (39).

Finally, with the help of (17) and (37) one obtains the coefficients for the multipole expansion of $\mathcal{U}^{\lambda'\lambda IM}_{\alpha'\alpha}$:

$$\mathcal{U}^{\lambda'\lambda IM}_{\alpha'\alpha} = \sum_{K,\,\kappa\in\kappa_X} \mathcal{U}^{\lambda'\lambda IM,\,K\kappa}_{\alpha'\alpha}\, d^K_{\lambda'-\lambda-M,\kappa}(\theta)\,. \tag{41}$$

Table 3. Listing of the sets κ_X determining the summation values κ in the multipole expansion (51) of a structure function for an observable $X=(\alpha'\alpha)$.

α'	0	3	0	3	1	0	2	0	3	1	3	2	1	2	1	2
α	0	0	3	3	0	1	0	2	1	3	2	3	1	1	2	2
κ_X	$\{0\}$				$\{-1,1\}$								$\{-2,0,2\}$			

The sets κ_X of the possible κ-values are listed in table 3 and the coefficients are given by

$$\mathcal{U}^{\lambda'\lambda IM,\,K\kappa}_{\alpha'\,\alpha} = \sum_{L'j'Lj} \mathcal{C}^{\lambda'\lambda IM,\,K}(L'j'Lj)\,\Omega^{\lambda'\lambda,\,K\kappa}_{\alpha'\alpha}(L'j'Lj) \tag{42}$$

where

$$\Omega^{\lambda'\lambda,\,K\kappa}_{\alpha'\alpha}(L'j'Lj) = \sum_{\mu'l's'\mu ls} \mathcal{D}^{K\kappa}_{\alpha'\alpha}(j'l's'jls)\,\mathcal{O}^{L'\lambda'\,*}(\mu'j'l's')\,\mathcal{O}^{L\lambda}(\mu jls)\,, \tag{43}$$

with

$$\mathcal{D}^{K\kappa}_{\alpha'\alpha}(j'l's'jls) = (-)^{l+s'+s}\,\hat{l}'\,\hat{l}\,\hat{s}'\,\hat{s} \times\sum_{\tau'\nu'\tau\nu}(-)^{\tau'+\tau}\,\hat\tau'\,\hat\tau\, s^{\tau'\nu'}_{\alpha'}\, s^{\tau\nu}_{\alpha}\left[\sum_S \hat S^2 \begin{pmatrix}\tau' & \tau & S\\ \nu' & \nu & -\kappa\end{pmatrix}\begin{Bmatrix}\frac12 & \frac12 & \tau'\\ \frac12 & \frac12 & \tau\\ s' & s & S\end{Bmatrix}\right.$$
$$\left.\times\left[\sum_{K'}\hat K'^2\begin{pmatrix}S & K & K'\\ \kappa & -\kappa & 0\end{pmatrix}\begin{pmatrix}K' & l & l'\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}S & K & K'\\ s & j & l\\ s' & j' & l'\end{Bmatrix}\right]\right]. \tag{44}$$

Similar expressions have been obtained by Raskin and Donnelly [12] for two cases: i) differential cross-section with beam and target polarization, and ii) polarization of one outgoing nucleon without consideration of target polarization (see eqs. (2.94) and (2.95) of [12]). Our result is a generalization to all polarization observables. Furthermore, our expressions are slightly different in their formal appearance, because we have separated explicitly the dependence on the target orientation angles from the angular dependence of the outgoing nucleons according to (9).

We would like to point out that the factorization in (42) is a manifestation of two ingredients. Firstly, angular-momentum selection rules (Wigner-Eckart theorem) and coupling schemes connected with the e.m. multipoles are contained in the coefficient $\mathcal{C}^{\lambda'\lambda IM,\,K}(L'j'Lj)$. This part is independent of the choice of the representation for the partial waves with good total angular momentum and also independent of the type of observable (α',α). On the other hand, the second factor $\Omega^{\lambda'\lambda,\,K\kappa}_{\alpha'\alpha}(L'j'Lj)$ reflects the structure of the spin operators of the final-state polarization as evaluated between the final-state partial waves, and thus depends on their representation. The Ω-coefficients for an uncoupled representation are given in appendix A.

The $\mathcal{D}$-coefficients have as symmetry properties under interchange $j'l's' \leftrightarrow jls$ and under complex conjugation

$$\mathcal{D}^{K\kappa}_{\alpha'\alpha}(jlsj'l's') = (-)^{j'+j+\delta^{(1)}_{(\alpha',\alpha)}}\,\mathcal{D}^{K-\kappa}_{\alpha'\alpha}(j'l's'jls)\,, \tag{45a}$$

$$\mathcal{D}^{K-\kappa}_{\alpha'\alpha}(j'l's'jls) = (-)^{l'+l+K+\delta^{(0)}_{(\alpha',\alpha)}+\delta^{(1)}_{(\alpha',\alpha)}}\,\mathcal{D}^{K\kappa}_{\alpha'\alpha}(j'l's'jls)\,, \tag{45b}$$

$$(\mathcal{D}^{K\kappa}_{\alpha'\alpha}(j'l's'jls))^* = (-)^{\delta^{(2)}_{(\alpha',\alpha)}}\,\mathcal{D}^{K\kappa}_{\alpha'\alpha}(j'l's'jls)\,, \tag{45c}$$

which follow straightforwardly using (22). Here, we have introduced as a shorthand

$$\delta^{(i)}_{(\alpha',\alpha)} = \delta_{\alpha',i}+\delta_{\alpha,i}\,. \tag{46}$$

With the help of (28), one can write $\mathcal{U}^{\lambda'\lambda IM}_{\alpha'\,\alpha}$ in a more compact form:

$$\mathcal{U}^{\lambda'\lambda IM,\,K\kappa}_{\alpha'\,\alpha} = 4\pi\, i^{\delta^{(2)}_{(\alpha',\alpha)}} \sum_{L'L\mu'j'\mu j}\mathcal{C}^{\lambda'\lambda IM,\,K}(L'j'Lj) \times\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu'j'\mu j)\,\widetilde{N}^{L'\,*}_{\lambda'}(\mu'j')\,\widetilde{N}^{L}_{\lambda}(\mu j)\,, \tag{47}$$

where $\widetilde{N}^L_\lambda(\mu j)$ incorporates the phase shift for convenience, *i.e.* $\widetilde{N}^L_\lambda(\mu j) = e^{i\delta^j_\mu}\,N^L_\lambda(\mu j)$, and

$$\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu'j'\mu j) = (-i)^{\delta^{(2)}_{(\alpha',\alpha)}}\sum_{l's'ls}\mathcal{D}^{K\kappa}_{\alpha'\alpha}(j'l's'jls)\,U^{j'}_{l's',\mu'}\,U^{j}_{ls,\mu}\,. \tag{48}$$

One should note the angular-momentum condition $|j'-j|\le K\le j'+j$.

Equation (47) is our central result. It allows one to organize the presentation of the coefficients of the multipole expansion in a very efficient way, because the dependencies on the initial state polarization d.o.f. (virtual-photon polarization and target) and on the multipolarities, contained in $\mathcal{C}^{\lambda'\lambda IM,\,K}(L'j'Lj)$, separate from the dependencies on the observable and the final-state quantum number μ, contained in $\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu'j'\mu j)$.

With the help of the symmetries of (39) and (45) one easily finds for $\widetilde{\mathcal{D}}$ the following symmetry properties:

$$\widetilde{\mathcal{D}}^{K-\kappa}_{\alpha'\alpha}(\mu'j'\mu j) = (-)^{K+\mu'+j'+\mu+j+\delta^{(0)}_{(\alpha',\alpha)}+\delta^{(1)}_{(\alpha',\alpha)}} \times\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu'j'\mu j)\,, \tag{49a}$$

Table 4. Listing of the values of $\beta(a)$ in the multipole expansion (51).

a	L/T	LT	TT
$\beta(a)$	0	1	2

$$\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu j\mu' j') = (-)^{j'+j+\delta^{(1)}_{(\alpha',\alpha)}}\,\widetilde{\mathcal{D}}^{K-\kappa}_{\alpha'\alpha}(\mu' j'\mu j) = \tag{49b}$$

$$(-)^{K+\mu'+\mu+\delta^{(0)}_{(\alpha',\alpha)}}\,\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu' j'\mu j)\,, \tag{49c}$$

$$\Big(\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu' j'\mu j)\Big)^* = \widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu' j'\mu j)\,. \tag{49d}$$

The symmetries of (39b) and (49b) allow one to derive a simple relation for the $\mathcal{U}$ under complex conjugation:

$$\Big(\mathcal{U}^{\lambda'\lambda IM,\,K\kappa}_{\alpha'\,\alpha}\Big)^* = (-)^{\lambda'+\lambda+\delta^{(1)}_{(\alpha',\alpha)}+\delta^{(2)}_{(\alpha',\alpha)}}\,\mathcal{U}^{\lambda\lambda' I-M,\,K-\kappa}_{\alpha'\,\alpha}\,. \tag{50}$$

This relation follows also directly from (23) using $(-)^\kappa = (-)^{\delta^{(1)}_{(\alpha',\alpha)}+\delta^{(2)}_{(\alpha',\alpha)}}$ (see table 3). From (50) follows that $\mathcal{U}^{\lambda\lambda I0,\,K0}_{\alpha'\,\alpha}$ is real or imaginary for $\lambda'=\lambda$, $M=0$ and $\kappa=0$, depending on whether $(-)^{\delta^{(1)}_{(\alpha',\alpha)}+\delta^{(2)}_{(\alpha',\alpha)}}$ is equal to 1 or -1, respectively.

Finally, one obtains the general expansion of a structure function in terms of the $d^K_{m'm}$ functions, which are related to the Jacobian polynomials in general, but for $m'=0$ or $m=0$ to the associated Legendre functions [19]

$$f^{(\prime)\,IM(\pm)}_a(X) = \sum_{K,\,\kappa\in\kappa_X} f^{(\prime)\,IM(\pm),\,K\,\kappa}_a(X)\, d^K_{-M-\beta(a),\kappa}(\theta)\,, \tag{51}$$

where $\beta(a)$ is listed in table 4, and the coefficients $f^{(\prime)\,IM(\pm),\,K\,\kappa}_a(X)$ are obtained via (13) from the foregoing multipole expansion (41). One should remember that an observable X is represented by $(\alpha'\alpha)$. Defining

$$\widetilde{\mathcal{C}}^{IM,\,K}_L(L'j'Lj) = \frac{8\,\pi}{1+\delta_{M,0}}\,\mathcal{C}^{00IM,\,K}(L'j'Lj) = \frac{32\sqrt{3}}{1+\delta_{M,0}}\,\pi\,\hat I\,\hat K^2\,(-)^L\,\hat j'\,\hat j\sum_J \hat J^2 \times\begin{pmatrix} J & I & K\\ 0 & M & -M\end{pmatrix}\begin{pmatrix} L' & L & J\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} j' & j & K\\ L' & L & J\\ 1 & 1 & I\end{Bmatrix}, \tag{52a}$$

$$\widetilde{\mathcal{C}}^{IM,\,K}_T(L'j'Lj) = \frac{16\,\pi}{1+\delta_{M,0}}\,\mathcal{C}^{11IM,\,K}(L'j'Lj) = -\frac{32\sqrt{3}}{1+\delta_{M,0}}\,\pi\,\hat I\,\hat K^2\,(-)^L\,\hat j'\,\hat j\sum_J \hat J^2 \times\begin{pmatrix} J & I & K\\ 0 & M & -M\end{pmatrix}\begin{pmatrix} L' & L & J\\ 1 & -1 & 0\end{pmatrix}\begin{Bmatrix} j' & j & K\\ L' & L & J\\ 1 & 1 & I\end{Bmatrix}, \tag{52b}$$

$$\begin{aligned}\widetilde{\mathcal{C}}^{IM\pm,\,K}_{LT}(L'j'Lj) &= \frac{16\,\pi}{1+\delta_{M,0}} \times\Big(\mathcal{C}^{01IM,\,K}(L'j'Lj) \pm (-)^{I+M}\mathcal{C}^{01I-M,\,K}(L'j'Lj)\Big)\\ &= \frac{32\sqrt{6}}{1+\delta_{M,0}}\,\pi\,\hat I\,\hat K^2\,(-)^L\,\hat j'\,\hat j\sum_J \hat J^2 \times\left[\begin{pmatrix} J & I & K\\ 1 & M & -M-1\end{pmatrix} \pm (-)^{I+M}\begin{pmatrix} J & I & K\\ 1 & -M & M-1\end{pmatrix}\right]\\ &\times\begin{pmatrix} L' & L & J\\ 0 & -1 & 1\end{pmatrix}\begin{Bmatrix} j' & j & K\\ L' & L & J\\ 1 & 1 & I\end{Bmatrix},\end{aligned} \tag{52c}$$

$$\begin{aligned}\widetilde{\mathcal{C}}^{IM\pm,\,K}_{TT}(L'j'Lj) &= \frac{8\,\pi}{1+\delta_{M,0}} \times\Big(\mathcal{C}^{-11IM,\,K}(L'j'Lj) \pm (-)^{I+M}\mathcal{C}^{-11I-M,\,K}(L'j'Lj)\Big)\\ &= -\frac{16\sqrt{3}}{1+\delta_{M,0}}\,\pi\,\hat I\,\hat K^2\,(-)^L\,\hat j'\,\hat j\sum_J \hat J^2 \times\left[\begin{pmatrix} J & I & K\\ 2 & M & -M-2\end{pmatrix} \pm (-)^{I+M}\begin{pmatrix} J & I & K\\ 2 & -M & M-2\end{pmatrix}\right]\\ &\times\begin{pmatrix} L' & L & J\\ -1 & -1 & 2\end{pmatrix}\begin{Bmatrix} j' & j & K\\ L' & L & J\\ 1 & 1 & I\end{Bmatrix},\end{aligned} \tag{52d}$$

one obtains in detail for the longitudinal- and transverse-structure functions

$$f^{IM,\,K\kappa}_L(X) = \sum_{L'\mu'j'L\mu j}\widetilde{\mathcal{C}}^{IM,\,K}_L(L'j'Lj) \times\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu' j'\mu j)\,\Re e\Big(i^{\bar\delta^X_I+\delta^{(2)}_{(\alpha',\alpha)}}\,\widetilde{C}^{L'*}(\mu' j')\,\widetilde{C}^{L}(\mu j)\Big)\,, \tag{53a}$$

$$f^{IM,\,K\kappa}_T(X) = \sum_{L'\mu'j'L\mu j}\widetilde{\mathcal{C}}^{IM,\,K}_T(L'j'Lj) \times\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu' j'\mu j)\,\Re e\Big(i^{\bar\delta^X_I+\delta^{(2)}_{(\alpha',\alpha)}}\,\widetilde{N}^{L'*}_1(\mu' j')\,\widetilde{N}^{L}_1(\mu j)\Big)\,, \tag{53b}$$

$$f'^{IM,\,K\kappa}_T(X) = -\sum_{L'\mu'j'L\mu j}\widetilde{\mathcal{C}}^{IM,\,K}_T(L'j'Lj) \times\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu' j'\mu j)\,\Im m\Big(i^{\bar\delta^X_I+\delta^{(2)}_{(\alpha',\alpha)}}\,\widetilde{N}^{L'*}_1(\mu' j')\,\widetilde{N}^{L}_1(\mu j)\Big)\,, \tag{53c}$$

and for the interference ones, distinguishing observables of type A

$$f^{IM\pm,\,K\kappa}_{TT}(X) = \sum_{L'\mu'j'L\mu j}\widetilde{\mathcal{C}}^{IM\pm,\,K}_{TT}(L'j'Lj) \times\widetilde{\mathcal{D}}^{K\kappa}_{\alpha'\alpha}(\mu' j'\mu j)\,\Re e\Big(i^{\bar\delta^X_I+\delta^{(2)}_{(\alpha',\alpha)}}\,\widetilde{N}^{L'*}_{-1}(\mu' j')\,\widetilde{N}^{L}_1(\mu j)\Big)\,, \tag{54a}$$

$$f_{LT}^{IM\pm,\,K\kappa}(X) = \sum_{L'\mu'j'L\mu j} \widetilde{\mathcal{C}}_{LT}^{IM\pm,\,K}(L'j'Lj) \times\widetilde{\mathcal{D}}_{\alpha'\alpha}^{K\kappa}(\mu'j'\mu j)\,\Re e\Big(i^{\bar{\delta}_I^X+\delta^{(2)}_{(\alpha',\alpha)}}\,\widetilde{C}^{L'*}(\mu'j')\,\widetilde{N}_1^L(\mu j)\Big)\,, \tag{54b}$$

$$f_{LT}^{\prime\,IM\pm,\,K\kappa}(X) = -\sum_{L'\mu'j'L\mu j} \widetilde{\mathcal{C}}_{LT}^{IM\pm,\,K}(L'j'Lj) \times\widetilde{\mathcal{D}}_{\alpha'\alpha}^{K\kappa}(\mu'j'\mu j)\,\Im m\Big(i^{\bar{\delta}_I^X+\delta^{(2)}_{(\alpha',\alpha)}}\,\widetilde{C}^{L'*}(\mu'j')\,\widetilde{N}_1^L(\mu j)\Big)\,, \tag{54c}$$

and observables of type B

$$f_{TT}^{IM\pm,\,K\kappa}(X) = \sum_{L'\mu'j'L\mu j} \widetilde{\mathcal{C}}_{TT}^{IM\mp,\,K}(L'j'Lj) \times\widetilde{\mathcal{D}}_{\alpha'\alpha}^{K\kappa}(\mu'j'\mu j)\,\Re e\Big(i^{\bar{\delta}_I^X+\delta^{(2)}_{(\alpha',\alpha)}}\,\widetilde{N}_{-1}^{L'*}(\mu'j')\,\widetilde{N}_1^L(\mu j)\Big)\,, \tag{55a}$$

$$f_{LT}^{IM\pm,\,K\kappa}(X) = \sum_{L'\mu'j'L\mu j} \widetilde{\mathcal{C}}_{LT}^{IM\mp,\,K}(L'j'Lj) \times\widetilde{\mathcal{D}}_{\alpha'\alpha}^{K\kappa}(\mu'j'\mu j)\,\Re e\Big(i^{\bar{\delta}_I^X+\delta^{(2)}_{(\alpha',\alpha)}}\,\widetilde{C}^{L'*}(\mu'j')\,\widetilde{N}_1^L(\mu j)\Big)\,, \tag{55b}$$

$$f_{LT}^{\prime\,IM\pm,\,K\kappa}(X) = -\sum_{L'\mu'j'L\mu j} \widetilde{\mathcal{C}}_{LT}^{IM\mp,\,K}(L'j'Lj) \times\widetilde{\mathcal{D}}_{\alpha'\alpha}^{K\kappa}(\mu'j'\mu j)\,\Im m\Big(i^{\bar{\delta}_I^X+\delta^{(2)}_{(\alpha',\alpha)}}\,\widetilde{C}^{L'*}(\mu'j')\,\widetilde{N}_1^L(\mu j)\Big)\,. \tag{55c}$$

With this we will conclude the present work. In appendix B we list more explicit expressions for the unpolarized differential cross-section. Furthermore, we have established a "mathematica" program, which allows one to evaluate explicitly the coefficients $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ for any observable up to a given maximal multipolarity $L_{\max}$ from which one can obtain the explicit contributions of the various multipole moments to a specific coefficient of the expansions (53) through (55). As an example we list in appendix C the coefficients $\widetilde{\mathcal{C}}$ up to $L_{\max}=3$ and $\widetilde{\mathcal{D}}$ for the differential cross-section up to $j_{\max}=4$. Upon request the authors will provide these coefficients for other observables and multipolarities.

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Appendix A. Multipole expansion for an uncoupled representation

We start from the scattering-wave analogous to (25) in the uncoupled representation [18]

$$|\vec{p}\,\lambda_p\,\lambda_n\rangle^{(-)} = \frac{1}{\sqrt{4\,\pi}}\sum_{jm_j}\hat{\jmath}\,|p\,j\,m_j;\lambda_p\lambda_n\rangle\,D^j_{m_j\lambda_{pn}}(R)\,, \tag{A.1}$$

where $\lambda_{pn}=\lambda_p+\lambda_n$, and $|p\,j\,m_j;\lambda_p\lambda_n\rangle$ denotes a partial wave with good angular momentum j and projection m_j on the photon momentum. It is defined by [18]

$$|p\,j\,m_j;\lambda_p\lambda_n\rangle = \frac{4\,\pi^{3/2}}{\hat{\jmath}} \times\int \mathrm{d}(\alpha\beta\gamma)\,D^{j\,*}_{m_j,\lambda_{pn}}(\alpha,\beta,\gamma)\,R_{\alpha\beta\gamma}|\vec{p}\,\lambda_p\,\lambda_n\rangle^{(-)}_{(\theta=0,\phi=0)}\,, \tag{A.2}$$

where $R_{\alpha\beta\gamma}$ denotes the rotation operator through the Euler angles (α,β,γ). Then the multipole expansion of the t-matrix reads

$$\begin{aligned} t_{\lambda_p\lambda_n\lambda\lambda_d}(\theta) =& (-)^{\lambda}\sqrt{1+\delta_{\lambda,0}} \times\sum_{Ljm_j}(1\lambda_d L\lambda|jm_j)\mathcal{O}^{L\lambda}(j\lambda_p\lambda_n)d^j_{m_j\lambda_{pn}}(\theta) = \\ & (-)^{1+\lambda}\sqrt{1+\delta_{\lambda,0}}\sum_{Ljm_j}(-)^{L+m_j} \times\begin{pmatrix}1 & L & j\\ \lambda_d & \lambda & -m_j\end{pmatrix}\mathcal{O}^{L\lambda}(j\lambda_p\lambda_n)d^j_{m_j\lambda_{pn}}(\theta)\,, \end{aligned} \tag{A.3}$$

with

$$\mathcal{O}^{L\lambda}(j\lambda_p\lambda_n) = \sqrt{4\pi}\,\Big[\delta_{|\lambda|,1}\Big(E^L(j\lambda_p\lambda_n)+\lambda M^L(j\lambda_p\lambda_n)\Big) +\delta_{\lambda,0}C^L(j\lambda_p\lambda_n)\Big]\,, \qquad (A.4)$$

where $E^L(j\lambda_p\lambda_n)$, $M^L(j\lambda_p\lambda_n)$ and $C^L(j\lambda_p\lambda_n)$ denote the reduced electric, magnetic and charge multipole matrix elements, respectively, between the deuteron state and the final-state partial wave $|p\,j\,m_j;\lambda_p\lambda_n\rangle$. Evaluating with this form of the t-matrix

$$\mathcal{U}^{\lambda'\lambda IM}_{\alpha'\alpha} = \sum_{\lambda'_p\lambda'_n\lambda'_d\lambda_p\lambda_n\lambda_d} t^*_{\lambda'_p\lambda'_n\lambda'\lambda'_d}\langle\lambda'_p|\sigma_{\alpha'}(p)|\lambda_p\rangle \times\langle\lambda'_n|\sigma_\alpha(n)|\lambda_n\rangle t_{\lambda_p\lambda_n\lambda\lambda_d}\left\langle\lambda_d\left|\tau^{[I]}_M\right|\lambda'_d\right\rangle, \quad (A.5)$$

one finds again (41) with the coefficient of the form (42). However, the explicit form of $\Omega^{\lambda'\lambda,\,K\kappa}_{\alpha'\alpha}(L'j'Lj)$ is now given by

$$\Omega^{\lambda'\lambda,\,K\kappa}_{\alpha'\alpha}(L'j'Lj) = (-)^{1+j+K}\sum_{\tau'\nu'\tau\nu}\hat\tau'\,\hat\tau\,s^{\tau'\nu'}_{\alpha'}\,s^{\tau\nu}_{\alpha} \times\sum_{\lambda'_p\lambda'_n\lambda_p\lambda_n}\begin{pmatrix}\frac12 & \tau' & \frac12\\ -\lambda'_p & \nu' & \lambda_p\end{pmatrix}\begin{pmatrix}\frac12 & \tau & \frac12\\ -\lambda'_n & \nu & \lambda_n\end{pmatrix} \times\begin{pmatrix}j' & j & K\\ \lambda'_{pn} & -\lambda_{pn} & -\kappa\end{pmatrix}\mathcal{O}^{L'\lambda'}(j'\lambda'_p\lambda'_n)^*\,\mathcal{O}^{L\lambda}(j\lambda_p\lambda_n)\,. \qquad (A.6)$$

This result is quite general and is also valid for a covariant description. The further evaluation will depend on the specific properties of the partial waves of good angular momentum chosen in a given dynamical approach.

For example, if one introduces the ls-representation according to [18] by

$$|p\,(ls)j\,m_j\rangle = \frac{1}{\hat\jmath}\sum_{\lambda_p\lambda_n\lambda}\hat l\left(\frac12\lambda_p\frac12\lambda_n|s\lambda_{pn}\right) \times(l0s\lambda|j\lambda)\,|p\,jm_j;\lambda_p\lambda_n\rangle\,, \qquad (A.7)$$

yielding by inversion

$$|p\,j\,m_j;\lambda_p\lambda_n\rangle = \frac{1}{\hat\jmath}\sum_{ls}\hat l\left(\frac12\lambda_p\frac12\lambda_n|s\lambda_{pn}\right) \times(l0s\lambda_{pn}|j\lambda_{pn})\,|p\,(ls)jm_j\,\rangle\,, \qquad (A.8)$$

and

$$\mathcal{O}^{L\lambda}(j\lambda_p\lambda_n) = \frac{1}{\hat\jmath}\sum_{ls}\hat l\left(\frac12\lambda_p\frac12\lambda_n|s\lambda_{pn}\right) \times(l0s\lambda_{pn}|j\lambda_{pn})\,\mathcal{O}^{L\lambda}(jls)\,, \qquad (A.9)$$

one recovers an expression analogous to the one in (43), *i.e.*

$$\Omega^{\lambda'\lambda,\,K\kappa}_{\alpha'\alpha}(L'j'Lj) = \sum_{l's'ls}\mathcal{D}^{K\kappa}_{\alpha'\alpha}(j'l's'jls) \times\mathcal{O}^{L'\lambda'}(j'l's')^*\,\mathcal{O}^{L\lambda}(jls)\,, \quad (A.10)$$

where $\mathcal{D}^{K\kappa}_{\alpha'\alpha}(j'l's'jls)$ is given in (44).

Appendix B. Multipole expansion of the form factors and structure functions of the differential cross-section

In order to obtain more explicit expressions for the multipole expansion of the structure functions of the differential cross-section

$$f^{(\prime)\,IM(\pm)}_a = \sum_K f^{(\prime)\,IM(\pm),\,K}_a\,d^K_{-M-\beta(a),0}(\theta)\,, \qquad (B.1)$$

using a simplified notation since in this case $\kappa = 0$ and thus is left out, we specialize (53) and (54) to $X = (\alpha'\alpha) = (00)$ and find

$$f^{IM,\,K}_L = \frac14\sum_{L'\mu'j'L\mu j}\Big((-)^{L'+\mu'+j'}+1\Big)\Big((-)^{L+\mu+j}+1\Big) \times\widetilde{\mathcal{C}}^{IM,\,K}_L(L'j'Lj)\,\widetilde{\mathcal{D}}^{K0}_{00}(\mu'j'\mu j) \times\Re e\Big(i^{\delta_{I,1}}\widetilde{C}^{L'*}(\mu'j')\,\widetilde{C}^L(\mu j)\Big)\,, \qquad (B.2a)$$

$$f^{IM,\,K}_T = \sum_{L'\mu'j'L\mu j}\widetilde{\mathcal{C}}^{IM,\,K}_T(L'j'Lj)\,\widetilde{\mathcal{D}}^{K0}_{00}(\mu'j'\mu j) \times\Re e\Big(i^{\delta_{I,1}}\widetilde{N}^{L'*}_1(\mu'j')\,\widetilde{N}^L_1(\mu j)\Big)\,, \qquad (B.2b)$$

$$f^{IM\pm,\,K}_{LT} = \frac12\sum_{L'\mu'j'L\mu j}\Big((-)^{L'+\mu'+j'}+1\Big)\widetilde{\mathcal{C}}^{IM,\,K}_{LT}(L'j'Lj) \times\widetilde{\mathcal{D}}^{K0}_{00}(\mu'j'\mu j)\,\Re e\Big(i^{\delta_{I,1}}\widetilde{C}^{L'*}(\mu'j')\,\widetilde{N}^L_1(\mu j)\Big)\,, \qquad (B.2c)$$

$$f^{IM\pm,\,K}_{TT} = \sum_{L'\mu'j'L\mu j}\widetilde{\mathcal{C}}^{IM,\,K}_{TT}(L'j'Lj)\,\widetilde{\mathcal{D}}^{K0}_{00}(\mu'j'\mu j) \times\Re e\Big(i^{\delta_{I,1}}\widetilde{N}^{L'*}_{-1}(\mu'j')\,\widetilde{N}^L_1(\mu j)\Big)\,, \qquad (B.2d)$$

$$f'^{IM,\,K}_T = -\sum_{L'\mu'j'L\mu j}\widetilde{\mathcal{C}}^{IM,\,K}_T(L'j'Lj) \times\widetilde{\mathcal{D}}^{K0}_{00}(\mu'j'\mu j)\,\Im m\Big(i^{\delta_{I,1}}\widetilde{N}^{L'*}_1(\mu'j')\,\widetilde{N}^L_1(\mu j)\Big)\,, \qquad (B.2e)$$

$$f'^{IM\pm,\,K}_{LT} = -\frac12\sum_{L'\mu'j'L\mu j}\Big((-)^{L'+\mu'+j'}+1\Big)\widetilde{\mathcal{C}}^{IM,\,K}_{LT}(L'j'Lj) \times\widetilde{\mathcal{D}}^{K0}_{00}(\mu'j'\mu j)\,\Im m\Big(i^{\delta_{I,1}}\widetilde{C}^{L'*}(\mu'j')\,\widetilde{N}^L_1(\mu j)\Big)\,, \qquad (B.2f)$$

where

$$\widetilde{\mathcal{D}}^{K0}_{00}(\mu'j'\mu j) = \frac{(-)^{j'}}{2}\sum_{l'ls}(-)^s\,\hat l'\,\hat l\begin{pmatrix}K & l & l'\\ 0 & 0 & 0\end{pmatrix} \times\begin{Bmatrix}j & l & s\\ l' & j' & K\end{Bmatrix}U^{j'}_{l's,\mu'}\,U^j_{ls,\mu}\,. \qquad (B.3)$$

Note that we have included the selection rule (30) for the Coulomb matrix elements and that the tilde over the multipole matrix elements indicates the incorporation of the hadronic-phase factor $e^{i\delta^j_\mu}$.

Now we specialize further in order to obtain the angular coefficients for the unpolarized differential cross-section

$$S_0 = c(k_1^{\rm lab},\, k_2^{\rm lab}) \sum_K \Big(\Big[\rho_L f_L^K + \rho_T f_T^K\Big] d_{00}^K(\theta) + \rho_{LT} f_{LT}^K\, d_{-10}^K(\theta)\, \cos\phi + \rho_{TT} f_{TT}^K\, d_{-20}^K(\theta)\, \cos 2\phi \Big)\,, \tag{B.4}$$

by setting $I = M = 0$ in (B.2) and evaluating $\widetilde{\mathcal{C}}_a^{00,\,K}(L'j'Lj)$ in (52) with

$$\mathcal{C}^{\lambda'\lambda 00,\,K}(L'j'Lj) = (-)^{\lambda+L'+L+j+1} \times 2\sqrt{3\,(1+\delta_{\lambda',0})(1+\delta_{\lambda,0})}\,\hat{K}^2\,\hat{j}'\,\hat{j} \times \begin{pmatrix} L' & L & K \\ \lambda' & -\lambda & \lambda-\lambda' \end{pmatrix} \begin{Bmatrix} j' & j & K \\ L & L' & 1 \end{Bmatrix}. \tag{B.5}$$

Writing for simplicity $f_{L/T}^K$ and $f_{LT/TT}^{(\prime)\,K}$ instead of $f_{L/T}^{00,\,K}$ and $f_{LT/TT}^{(\prime)\,00+,\,K}$, respectively, one obtains

$$f_L^K = -4\,\pi\,\hat{K}^2 \sum_{L'\mu'j'L\mu j} (-)^{L'+L+j}\,\hat{j}'\,\hat{j} \times \begin{pmatrix} L' & L & K \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} j' & j & K \\ L & L' & 1 \end{Bmatrix} \times\Big((-)^{L'+\mu'+j'}+1\Big)\Big((-)^{L+\mu+j}+1\Big) \times \widetilde{\mathcal{D}}_{00}^{K0}(\mu'j'\mu j)\,\Re e\Big(\widetilde{C}^{L'*}(\mu'j')\,\widetilde{C}^{L}(\mu j)\Big), \tag{B.6a}$$

$$f_T^K = 16\,\pi\,\hat{K}^2 \sum_{L'\mu'j'L\mu j} (-)^{L'+L+j}\,\hat{j}'\,\hat{j} \times \begin{pmatrix} L' & L & K \\ 1 & -1 & 0 \end{pmatrix} \begin{Bmatrix} j' & j & K \\ L & L' & 1 \end{Bmatrix} \widetilde{\mathcal{D}}_{00}^{K0}(\mu'j'\mu j) \times \Re e\Big(\widetilde{N}_1^{L'*}(\mu'j')\,\widetilde{N}_1^{L}(\mu j)\Big)\,, \tag{B.6b}$$

$$f_{LT}^K = 16\sqrt{2}\,\pi\,\hat{K}^2 \sum_{L'\mu'j'L\mu j} (-)^{L'+L+j}\,\hat{j}'\,\hat{j} \times \begin{pmatrix} L' & L & K \\ 0 & -1 & 1 \end{pmatrix} \begin{Bmatrix} j' & j & K \\ L & L' & 1 \end{Bmatrix} \times\Big((-)^{L'+\mu'+j'}+1\Big)\widetilde{\mathcal{D}}_{00}^{K0}(\mu'j'\mu j) \times \Re e\Big(\widetilde{C}^{L'*}(\mu'j')\,\widetilde{N}_1^{L}(\mu j)\Big)\,, \tag{B.6c}$$

$$f_{TT}^K = 16\,\pi\,\hat{K}^2 \sum_{L'\mu'j'L\mu j} (-)^{L'+L+j}\,\hat{j}'\,\hat{j} \times \begin{pmatrix} L' & L & K \\ -1 & -1 & 2 \end{pmatrix} \begin{Bmatrix} j' & j & K \\ L & L' & 1 \end{Bmatrix} \widetilde{\mathcal{D}}_{00}^{K0}(\mu'j'\mu j) \times \Re e\Big(\widetilde{N}_{-1}^{L'*}(\mu'j')\,\widetilde{N}_1^{L}(\mu j)\Big)\,. \tag{B.6d}$$

Finally, we will give the explicit multipole decomposition of the various inclusive form factors. The form factors can be obtained from the $(K=0)$-coefficients of (B.2) according to

$$F_a^{(\prime)I-M} = (-)^{I+M}\,(1+\delta_{M,0})\,\frac{\pi}{3}\,\big(f_a^{(\prime)IM+,\,0} - f_a^{(\prime)IM-,\,0}\big)\,. \tag{B.7}$$

Using

$$\widetilde{\mathcal{D}}_{00}^{00}(\mu'j'\mu j) = \frac{1}{2\,\hat{j}}\delta_{\mu',\mu}\,\delta_{j',j}\,, \tag{B.8}$$

and

$$\mathcal{C}^{\lambda'\lambda IM,\,0}(L'j'Lj) = \delta_{j',j}\,\delta_{M,\lambda'-\lambda}\,(-)^{1+j+\lambda}\,2\,\hat{I}\,\hat{j}\sqrt{3\,(1+\delta_{\lambda',0})(1+\delta_{\lambda,0})} \times \begin{pmatrix} L' & L & I \\ \lambda' & -\lambda & \lambda-\lambda' \end{pmatrix} \begin{Bmatrix} L' & L & I \\ 1 & 1 & j \end{Bmatrix}, \tag{B.9}$$

one finds for $K=0$ from (47)

$$\mathcal{U}_{00}^{\lambda'\lambda IM,\,00} = (-)^{\lambda+1}\,\delta_{M,\lambda'-\lambda}\,4\,\pi\,\hat{I}\,\sqrt{3\,(1+\delta_{\lambda,0})(1+\delta_{\lambda',0})} \times \sum_{L'L\mu j} (-)^j \begin{pmatrix} L & L' & I \\ \lambda & -\lambda' & \lambda'-\lambda \end{pmatrix} \times \begin{Bmatrix} L & L' & I \\ 1 & 1 & j \end{Bmatrix} e^{-2\rho_\mu^j} N_{\lambda'}^{L'}(\mu j)^*\, N_\lambda^L(\mu j). \tag{B.10}$$

One should note that in the foregoing equation the real part of the hadronic phases of the final states has disappeared because the multipole transitions L' and L lead to the same state $|\mu j\rangle$, whereas the inelasticity, denoted by ρ_μ^j, remains.

Formally one finds nonvanishing contributions for $M = \lambda' - \lambda$ only, *i.e.*, $f_{L/T}^{I0,0}$, $f_{LT}^{(\prime)\,I-1\pm,\,0}$ for $I = 1, 2$, $f_T'^{\,10,0}$ and $f_{TT}^{2-2\pm,\,0}$. According to (B.2), they are the real or imaginary parts of products of multipole matrix elements. As already mentioned, in the case of time reversal invariance, these matrix elements can be made real by a proper choice of phase convention below the pion production threshold. Therefore, those contributions involving the imaginary part vanish below the pion production threshold. This refers to $f_{LT}^{1-1\pm,\,0}$ and $f_{LT}'^{\,2-1\pm,\,0}$. If, however, one considers the *np*-channel above the pion threshold without explicit consideration of the $NN\pi$-channels in a coupled-channel approach including isobar degrees of freedom with complex propagators, the multipole matrix elements cannot all be made real. In this case, the latter two form factors become nonvanishing [1]. Whether this is an artefact of such an approach is an open question.

Now we will list the multipole expansion of those form factors of $d(e,e')np$, which are nonvanishing below pion threshold. They have already been reported before in [1]. The unpolarized form factors are given by

$$F_L = \frac{16\pi^2}{3}\sum_{Lj\mu}\frac{e^{-2\rho_\mu^j}}{2L+1}|C^L(\mu j)|^2, \qquad \text{(B.11a)}$$

$$F_T = \frac{16\pi^2}{3}\sum_{Lj\mu}\frac{e^{-2\rho_\mu^j}}{2L+1} \times(|E^L(\mu j)|^2 + |M^L(\mu j)|^2)\,, \qquad \text{(B.11b)}$$

the vector polarization form factors by

$$F_{LT}'^{1-1} = 32\pi^2\sqrt{2}\sum_{LL'j\mu}(-)^j\begin{pmatrix}L' & L & 1\\ 0 & -1 & 1\end{pmatrix} \times\begin{Bmatrix}L' & L & 1\\ 1 & 1 & j\end{Bmatrix}e^{-2\rho_\mu^j}\,\Re e[C^{L'}(\mu j)^* N_1^L(\mu j)], \qquad \text{(B.11c)}$$

$$F_T'^{10} = 16\pi^2\sum_{LL'j\mu}(-)^j\begin{pmatrix}L' & L & 1\\ 1 & -1 & 0\end{pmatrix} \times\begin{Bmatrix}L' & L & 1\\ 1 & 1 & j\end{Bmatrix}e^{-2\rho_\mu^j}\,\Re e[N_1^{L'}(\mu j)^* N_1^L(\mu j)]\,, \qquad \text{(B.11d)}$$

and finally the tensor polarization form factors

$$F_L^{20} = -16\pi^2\sqrt{\frac{5}{3}}\sum_{LL'j\mu}(-)^j\begin{pmatrix}L' & L & 2\\ 0 & 0 & 0\end{pmatrix} \times\begin{Bmatrix}L' & L & 2\\ 1 & 1 & j\end{Bmatrix}e^{-2\rho_\mu^j}\,\Re e[C^{L'}(\mu j)^* C^L(\mu j)], \qquad \text{(B.12a)}$$

$$F_{LT}^{2-1} = 32\pi^2\sqrt{\frac{10}{3}}\sum_{LL'j\mu}(-)^j\begin{pmatrix}L' & L & 2\\ 0 & -1 & 1\end{pmatrix} \times\begin{Bmatrix}L' & L & 2\\ 1 & 1 & j\end{Bmatrix}e^{-2\rho_\mu^j}\,\Re e[C^{L'}(\mu j)^* N_1^L(\mu j)], \qquad \text{(B.12b)}$$

$$F_T^{20} = 16\pi^2\sqrt{\frac{5}{3}}\sum_{LL'j\mu}(-)^j\begin{pmatrix}L' & L & 2\\ 1 & -1 & 0\end{pmatrix} \times\begin{Bmatrix}L' & L & 2\\ 1 & 1 & j\end{Bmatrix}e^{-2\rho_\mu^j}\,\Re e[N_1^{L'}(\mu j)^* N_1^L(\mu j)], \qquad \text{(B.12c)}$$

$$F_{TT}^{2-2} = 16\pi^2\sqrt{\frac{5}{3}}\sum_{LL'j\mu}(-)^j\begin{pmatrix}L' & L & 2\\ -1 & -1 & 2\end{pmatrix} \times\begin{Bmatrix}L' & L & 2\\ 1 & 1 & j\end{Bmatrix}e^{-2\rho_\mu^j}\,\Re e[N_{-1}^{L'}(\mu j)^* N_1^L(\mu j)]. \qquad \text{(B.12d)}$$

Above the pion threshold, the following additional form factors appear in $d(e,e')np$ as has already been mentioned above.

$$F_{LT}^{1-1} = -32\pi^2\sqrt{2}\sum_{LL'j\mu}(-)^j\begin{pmatrix}L' & L & 1\\ 0 & -1 & 1\end{pmatrix} \times\begin{Bmatrix}L' & L & 1\\ 1 & 1 & j\end{Bmatrix}e^{-2\rho_\mu^j}\,\Im m[C^{L'}(\mu j)^* N_1^L(\mu j)]\,, \qquad \text{(B.13a)}$$

$$F_{LT}'^{2-1} = -32\pi^2\sqrt{\frac{10}{3}}\sum_{LL'j\mu}(-)^j\begin{pmatrix}L' & L & 2\\ 0 & -1 & 1\end{pmatrix} \times\begin{Bmatrix}L' & L & 2\\ 1 & 1 & j\end{Bmatrix}e^{-2\rho_\mu^j}\,\Im m[C^{L'}(\mu j)^* N_1^L(\mu j)]\,. \qquad \text{(B.13b)}$$

We would like to emphasize that if one considers the completely inclusive process $d(e,e')X$, the corresponding additional form factors will vanish as long as the time reversal invariance holds.

Appendix C. Listing of the coefficients $\widetilde{\mathcal{C}}$ up to $L_{max} = 3$ and $\widetilde{\mathcal{D}}$ for the unpolarized differential cross-section up to $j_{max} = 4$

As an example, we list here the coefficients $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ for the unpolarized differential cross-section. First we will consider the coefficients $\widetilde{\mathcal{C}}$ which simplify considerably for the case of no target polarization, *i.e.*, $I = 0$ and $M = 0$, and obtain

$$\widetilde{\mathcal{C}}_L^{00,K}(L'j'Lj) = 16\,\pi\,(-)^{K+j+1} \times\hat{j}'\,\hat{j}\,\hat{K}^2\begin{pmatrix}L' & L & K\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}j' & j & K\\ L & L' & 1\end{Bmatrix}, \qquad \text{(C.1a)}$$

$$\widetilde{\mathcal{C}}_T^{00,K}(L'j'Lj) = 16\,\pi\,(-)^{L'+L+j} \times\hat{j}'\,\hat{j}\,\hat{K}^2\begin{pmatrix}L' & L & K\\ 1 & -1 & 0\end{pmatrix}\begin{Bmatrix}j' & j & K\\ L & L' & 1\end{Bmatrix}, \qquad \text{(C.1b)}$$

$$\widetilde{\mathcal{C}}_{LT}^{00,K}(L'j'Lj) = 32\,\pi\,\sqrt{2}(-)^{L'+L+j} \times\hat{j}'\,\hat{j}\,\hat{K}^2\begin{pmatrix}L' & L & K\\ 0 & -1 & 1\end{pmatrix}\begin{Bmatrix}j' & j & K\\ L & L' & 1\end{Bmatrix}, \qquad \text{(C.1c)}$$

$$\widetilde{\mathcal{C}}_{TT}^{00,K}(L'j'Lj) = 16\,\pi\,(-)^{L'+L+j} \times\hat{j}'\,\hat{j}\,\hat{K}^2\begin{pmatrix}L' & L & K\\ -1 & -1 & 2\end{pmatrix}\begin{Bmatrix}j' & j & K\\ L & L' & 1\end{Bmatrix}. \qquad \text{(C.1d)}$$

For $K = 0$, the only nonvanishing coefficients are

$$\widetilde{\mathcal{C}}_{L/T}^{00,0}(L'j'Lj) = 16\,\pi\,\frac{\hat{j}}{\hat{L}^2}\,\delta_{j',j}\,\delta_{L',L}\,. \qquad \text{(C.2)}$$

Table 5. $\widetilde{\mathcal{C}}_L^{00,K}(L'j'Lj)$ for $L_{\max}=3$ and $j_{\max}=4$.

$(L'j'Lj)$	$\widetilde{\mathcal{C}}_L$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_L$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_L$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_L$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_L$
				$K=0$					
(1010)	$\frac{16}{3}\pi$	(0101)	$16\sqrt{3}\pi$	(1111)	$\frac{16}{\sqrt{3}}\pi$	(2121)	$\frac{16}{5}\sqrt{3}\pi$	(1212)	$\frac{16}{3}\sqrt{5}\pi$
(2222)	$\frac{16}{\sqrt{5}}\pi$	(3232)	$\frac{16}{7}\sqrt{5}\pi$	(2323)	$\frac{16}{5}\sqrt{7}\pi$	(3333)	$\frac{16}{\sqrt{7}}\pi$	(3434)	$\frac{48}{7}\pi$
				$K=1$					
(0110)	-16π	(2110)	$16\sqrt{\frac{2}{5}}\pi$	(1101)	$-16\sqrt{3}\pi$	(2111)	$-8\sqrt{\frac{6}{5}}\pi$	(1201)	$16\sqrt{5}\pi$
(1221)	$-\frac{8}{5}\sqrt{2}\pi$	(2211)	$24\sqrt{\frac{2}{5}}\pi$	(3221)	$\frac{144}{5\sqrt{7}}\pi$	(2212)	$-8\sqrt{\frac{6}{5}}\pi$	(3222)	$-16\sqrt{\frac{3}{35}}\pi$
(2312)	$\frac{16}{5}\sqrt{42}\pi$	(2332)	$-\frac{16}{35}\sqrt{3}\pi$	(3322)	$32\sqrt{\frac{6}{35}}\pi$	(3323)	$-16\sqrt{\frac{3}{35}}\pi$	(3423)	$\frac{144}{7}\sqrt{\frac{3}{5}}\pi$
				$K=2$					
(1111)	$8\sqrt{\frac{10}{3}}\pi$	(2101)	$16\sqrt{3}\pi$	(2121)	$-8\sqrt{\frac{6}{5}}\pi$	(1210)	$-\frac{16}{3}\sqrt{10}\pi$	(3210)	$16\sqrt{\frac{5}{7}}\pi$
(1211)	$-8\sqrt{10}\pi$	(2201)	$-16\sqrt{5}\pi$	(2221)	$-8\sqrt{2}\pi$	(3211)	$-16\sqrt{\frac{5}{7}}\pi$	(1212)	$-\frac{8}{3}\sqrt{70}\pi$
(2222)	$-8\sqrt{\frac{10}{7}}\pi$	(3212)	$\frac{16}{7}\sqrt{5}\pi$	(3232)	$-\frac{64}{7}\sqrt{\frac{10}{7}}\pi$	(2301)	$16\sqrt{7}\pi$	(2321)	$-16\sqrt{\frac{2}{35}}\pi$
(3311)	$16\sqrt{\frac{10}{7}}\pi$	(2322)	$-\frac{32}{\sqrt{7}}\pi$	(3312)	$-16\sqrt{\frac{5}{7}}\pi$	(3332)	$-\frac{16}{7}\sqrt{10}\pi$	(2323)	$-64\sqrt{\frac{3}{35}}\pi$
(3333)	$-8\sqrt{\frac{15}{7}}\pi$	(3412)	$\frac{48}{7}\sqrt{15}\pi$	(3432)	$-\frac{16}{7}\sqrt{\frac{10}{21}}\pi$	(3433)	$-\frac{40}{7}\sqrt{\frac{5}{3}}\pi$	(3434)	$-\frac{40}{7}\sqrt{\frac{55}{7}}\pi$
				$K=3$					
(1221)	$\frac{48}{5}\sqrt{7}\pi$	(2211)	$16\sqrt{\frac{7}{5}}\pi$	(3201)	$16\sqrt{5}\pi$	(3221)	$-\frac{32}{5}\sqrt{2}\pi$	(2212)	$16\sqrt{\frac{14}{5}}\pi$
(3222)	$\frac{32}{\sqrt{5}}\pi$	(2310)	$-16\sqrt{\frac{7}{5}}\pi$	(2311)	$-16\sqrt{\frac{14}{5}}\pi$	(3301)	$-16\sqrt{7}\pi$	(3321)	$-8\sqrt{\frac{14}{5}}\pi$
(2312)	$-\frac{16}{5}\sqrt{42}\pi$	(2332)	$\frac{64}{5\sqrt{3}}\pi$	(3322)	$-8\sqrt{\frac{14}{15}}\pi$	(3323)	$16\sqrt{\frac{14}{15}}\pi$	(3401)	48π
(3421)	$-8\sqrt{\frac{2}{5}}\pi$	(3422)	$-8\sqrt{\frac{10}{3}}\pi$	(3423)	$-16\sqrt{\frac{22}{15}}\pi$				
				$K=4$					
(2222)	$-96\sqrt{\frac{2}{35}}\pi$	(3212)	$-\frac{96}{7}\sqrt{5}\pi$	(3232)	$\frac{48}{7}\sqrt{\frac{10}{7}}\pi$	(2321)	$\frac{144}{5}\sqrt{\frac{6}{7}}\pi$	(3311)	$24\sqrt{\frac{6}{7}}\pi$
(2322)	$48\sqrt{\frac{2}{7}}\pi$	(3312)	$24\sqrt{\frac{10}{7}}\pi$	(3332)	$\frac{48}{7}\sqrt{5}\pi$	(2323)	$\frac{48}{5}\sqrt{\frac{22}{7}}\pi$	(3333)	$24\sqrt{\frac{2}{77}}\pi$
(3410)	$-32\sqrt{\frac{3}{7}}\pi$	(3411)	$-24\sqrt{\frac{10}{7}}\pi$	(3412)	$-\frac{8}{7}\sqrt{330}\pi$	(3432)	$\frac{144}{7}\sqrt{\frac{15}{77}}\pi$	(3433)	$\frac{72}{7}\sqrt{\frac{30}{11}}\pi$
(3434)	$\frac{216}{7}\sqrt{\frac{26}{77}}\pi$								
				$K=5$					
(2332)	$-\frac{80}{7}\sqrt{\frac{22}{3}}\pi$	(3322)	$-16\sqrt{\frac{55}{21}}\pi$	(3323)	$-16\sqrt{\frac{55}{21}}\pi$	(3421)	$16\sqrt{\frac{22}{7}}\pi$	(3422)	$16\sqrt{\frac{55}{21}}\pi$
(3423)	$\frac{16}{7}\sqrt{\frac{143}{3}}\pi$								
				$K=6$					
(3333)	$40\sqrt{\frac{39}{77}}\pi$	(3432)	$-\frac{160}{7}\sqrt{\frac{65}{33}}\pi$	(3433)	$-40\sqrt{\frac{13}{33}}\pi$	(3434)	$-\frac{40}{7}\sqrt{\frac{65}{11}}\pi$		

Table 6. $\widetilde{\mathcal{C}}_T^{00,K}(L'j'Lj)$ for $L_{\max}=3$.

$(L'j'Lj)$	$\widetilde{\mathcal{C}}_T$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_T$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_T$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_T$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_T$
				$K=0$					
(1010)	$\frac{16}{3}\pi$	(1111)	$\frac{16}{\sqrt{3}}\pi$	(2121)	$\frac{16}{5}\sqrt{3}\pi$	(1212)	$\frac{16}{3}\sqrt{5}\pi$	(2222)	$\frac{16}{\sqrt{5}}\pi$
(3232)	$\frac{16}{7}\sqrt{5}\pi$	(2323)	$\frac{16}{5}\sqrt{7}\pi$	(3333)	$\frac{16}{\sqrt{7}}\pi$	(3434)	$\frac{48}{7}\pi$		
				$K=1$					
(2110)	$8\sqrt{\frac{6}{5}}\pi$	(2111)	$-12\sqrt{\frac{2}{5}}\pi$	(1221)	$-\frac{4}{5}\sqrt{6}\pi$	(2211)	$12\sqrt{\frac{6}{5}}\pi$	(3221)	$\frac{96}{5}\sqrt{\frac{2}{7}}\pi$
(2212)	$-12\sqrt{\frac{2}{5}}\pi$	(3222)	$-32\sqrt{\frac{2}{105}}\pi$	(2312)	$\frac{24}{5}\sqrt{14}\pi$	(2332)	$-\frac{32}{35}\sqrt{\frac{2}{3}}\pi$	(3322)	$\frac{128}{\sqrt{105}}\pi$
(3323)	$-32\sqrt{\frac{2}{105}}\pi$	(3423)	$\frac{96}{7}\sqrt{\frac{6}{5}}\pi$						
				$K=2$					
(1111)	$-4\sqrt{\frac{10}{3}}\pi$	(2121)	$-4\sqrt{\frac{6}{5}}\pi$	(1210)	$\frac{8}{3}\sqrt{10}\pi$	(3210)	$16\sqrt{\frac{10}{21}}\pi$	(1211)	$4\sqrt{10}\pi$
(2221)	$-4\sqrt{2}\pi$	(3211)	$-16\sqrt{\frac{10}{21}}\pi$	(1212)	$\frac{4}{3}\sqrt{70}\pi$	(2222)	$-4\sqrt{\frac{10}{7}}\pi$	(3212)	$\frac{16}{7}\sqrt{\frac{10}{3}}\pi$
(3232)	$-\frac{48}{7}\sqrt{\frac{10}{7}}\pi$	(2321)	$-8\sqrt{\frac{2}{35}}\pi$	(3311)	$32\sqrt{\frac{5}{21}}\pi$	(2322)	$-\frac{16}{\sqrt{7}}\pi$	(3312)	$-16\sqrt{\frac{10}{21}}\pi$
(3332)	$-\frac{12}{7}\sqrt{10}\pi$	(2323)	$-32\sqrt{\frac{3}{35}}\pi$	(3333)	$-6\sqrt{\frac{15}{7}}\pi$	(3412)	$\frac{48}{7}\sqrt{10}\pi$	(3432)	$-\frac{4}{7}\sqrt{\frac{30}{7}}\pi$
(3433)	$-\frac{10}{7}\sqrt{15}\pi$	(3434)	$-\frac{30}{7}\sqrt{\frac{55}{7}}\pi$						
				$K=3$					
(1221)	$-\frac{16}{5}\sqrt{21}\pi$	(2211)	$-16\sqrt{\frac{7}{15}}\pi$	(3221)	$-\frac{16}{5}\pi$	(2212)	$-16\sqrt{\frac{14}{15}}\pi$	(3222)	$8\sqrt{\frac{2}{5}}\pi$
(2310)	$16\sqrt{\frac{7}{15}}\pi$	(2311)	$16\sqrt{\frac{14}{15}}\pi$	(3321)	$-4\sqrt{\frac{7}{5}}\pi$	(2312)	$\frac{16}{5}\sqrt{14}\pi$	(2332)	$\frac{16}{5}\sqrt{\frac{2}{3}}\pi$
(3322)	$-4\sqrt{\frac{7}{15}}\pi$	(3323)	$8\sqrt{\frac{7}{15}}\pi$	(3421)	$-\frac{4}{\sqrt{5}}\pi$	(3422)	$-4\sqrt{\frac{5}{3}}\pi$	(3423)	$-8\sqrt{\frac{11}{15}}\pi$
				$K=4$					
(2222)	$64\sqrt{\frac{2}{35}}\pi$	(3212)	$\frac{24}{7}\sqrt{30}\pi$	(3232)	$\frac{8}{7}\sqrt{\frac{10}{7}}\pi$	(2321)	$-\frac{96}{5}\sqrt{\frac{6}{7}}\pi$	(3311)	$-\frac{36}{\sqrt{7}}\pi$
(2322)	$-32\sqrt{\frac{2}{7}}\pi$	(3312)	$-12\sqrt{\frac{15}{7}}\pi$	(3332)	$\frac{8}{7}\sqrt{5}\pi$	(2323)	$-\frac{32}{5}\sqrt{\frac{22}{7}}\pi$	(3333)	$4\sqrt{\frac{2}{77}}\pi$
(3410)	$24\sqrt{\frac{2}{7}}\pi$	(3411)	$12\sqrt{\frac{15}{7}}\pi$	(3412)	$\frac{12}{7}\sqrt{55}\pi$	(3432)	$\frac{24}{7}\sqrt{\frac{15}{77}}\pi$	(3433)	$\frac{12}{7}\sqrt{\frac{30}{11}}\pi$
(3434)	$\frac{36}{7}\sqrt{\frac{26}{77}}\pi$								
				$K=5$					
(2332)	$\frac{80}{7}\sqrt{\frac{11}{3}}\pi$	(3322)	$8\sqrt{\frac{110}{21}}\pi$	(3323)	$8\sqrt{\frac{110}{21}}\pi$	(3421)	$-16\sqrt{\frac{11}{7}}\pi$	(3422)	$-8\sqrt{\frac{110}{21}}\pi$
(3423)	$-\frac{8}{7}\sqrt{\frac{286}{3}}\pi$								
				$K=6$					
(3333)	$-30\sqrt{\frac{39}{77}}\pi$	(3432)	$\frac{40}{7}\sqrt{\frac{195}{11}}\pi$	(3433)	$10\sqrt{\frac{39}{11}}\pi$	(3434)	$\frac{30}{7}\sqrt{\frac{65}{11}}\pi$		

Table 7. $\widetilde{\mathcal{C}}_{LT}^{001,K}(L'j'Lj)$ for $L_{\max}=3$.

$(L'j'Lj)$	$\widetilde{\mathcal{C}}_{LT}$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_{LT}$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_{LT}$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_{LT}$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_{LT}$
				$K=1$					
(1021)	$-16\sqrt{\frac{3}{5}}\,\pi$	(0110)	$-16\sqrt{2}\,\pi$	(2110)	$-\frac{16}{\sqrt{5}}\,\pi$	(0111)	$-16\sqrt{6}\,\pi$	(1121)	$-\frac{24}{\sqrt{5}}\,\pi$
(2111)	$8\sqrt{\frac{3}{5}}\,\pi$	(0112)	$-16\sqrt{10}\,\pi$	(1122)	$-24\sqrt{\frac{3}{5}}\,\pi$	(2112)	$-\frac{8}{5}\,\pi$	(2132)	$-\frac{96}{5}\sqrt{\frac{3}{7}}\,\pi$
(1221)	$-\frac{8}{5}\sqrt{3}\,\pi$	(2211)	$-\frac{24}{\sqrt{5}}\,\pi$	(3221)	$-\frac{48}{5}\sqrt{\frac{6}{7}}\,\pi$	(1222)	$-\frac{24}{\sqrt{5}}\,\pi$	(2212)	$8\sqrt{\frac{3}{5}}\,\pi$
(2232)	$-\frac{32}{\sqrt{35}}\,\pi$	(3222)	$16\sqrt{\frac{2}{35}}\,\pi$	(1223)	$-\frac{48}{5}\sqrt{7}\,\pi$	(2233)	$-64\sqrt{\frac{2}{35}}\,\pi$	(3223)	$-\frac{16}{35}\sqrt{2}\,\pi$
(2312)	$-\frac{16}{5}\sqrt{21}\,\pi$	(2332)	$-\frac{32}{35}\,\pi$	(3322)	$-\frac{64}{\sqrt{35}}\,\pi$	(2333)	$-\frac{32}{\sqrt{35}}\,\pi$	(3323)	$16\sqrt{\frac{2}{35}}\,\pi$
(2334)	$-\frac{288}{7\sqrt{5}}\,\pi$	(3423)	$-\frac{144}{7}\sqrt{\frac{2}{5}}\,\pi$						
				$K=2$					
(1012)	$-16\sqrt{\frac{5}{3}}\,\pi$	(1032)	$\frac{64}{3}\sqrt{\frac{5}{7}}\,\pi$	(0121)	$16\sqrt{6}\,\pi$	(1111)	$8\sqrt{5}\,\pi$	(2121)	$-8\sqrt{\frac{3}{5}}\,\pi$
(0122)	$16\sqrt{10}\,\pi$	(1112)	$8\sqrt{15}\,\pi$	(1132)	$\frac{64}{3}\sqrt{\frac{5}{7}}\,\pi$	(2122)	$8\,\pi$	(0123)	$16\sqrt{14}\,\pi$
(1133)	$\frac{64}{3}\sqrt{\frac{10}{7}}\,\pi$	(2123)	$-\frac{16}{\sqrt{35}}\,\pi$	(1210)	$-16\sqrt{\frac{5}{3}}\,\pi$	(3210)	$-16\sqrt{\frac{10}{21}}\,\pi$	(1211)	$-8\sqrt{15}\,\pi$
(2221)	$-8\,\pi$	(3211)	$16\sqrt{\frac{10}{21}}\,\pi$	(1212)	$-8\sqrt{\frac{35}{3}}\,\pi$	(1232)	$\frac{64}{21}\sqrt{5}\,\pi$	(2222)	$-8\sqrt{\frac{5}{7}}\,\pi$
(3212)	$-\frac{16}{7}\sqrt{\frac{10}{3}}\,\pi$	(3232)	$-\frac{32}{7}\sqrt{\frac{10}{7}}\,\pi$	(1233)	$\frac{64}{3}\sqrt{\frac{5}{7}}\,\pi$	(2223)	$16\sqrt{\frac{2}{7}}\,\pi$	(3233)	$\frac{8}{7}\sqrt{10}\,\pi$
(1234)	$\frac{64}{7}\sqrt{15}\,\pi$	(3234)	$-\frac{8}{7}\sqrt{\frac{10}{21}}\,\pi$	(2321)	$-\frac{16}{\sqrt{35}}\,\pi$	(3311)	$-32\sqrt{\frac{5}{21}}\,\pi$	(2322)	$-16\sqrt{\frac{2}{7}}\,\pi$
(3312)	$16\sqrt{\frac{10}{21}}\,\pi$	(3332)	$-\frac{8}{7}\sqrt{10}\,\pi$	(2323)	$-32\sqrt{\frac{6}{35}}\,\pi$	(3333)	$-4\sqrt{\frac{15}{7}}\,\pi$	(3334)	$\frac{20}{7}\sqrt{\frac{5}{3}}\,\pi$
(3412)	$-\frac{48}{7}\sqrt{10}\,\pi$	(3432)	$-\frac{8}{7}\sqrt{\frac{10}{21}}\,\pi$	(3433)	$-\frac{20}{7}\sqrt{\frac{5}{3}}\,\pi$	(3434)	$-\frac{20}{7}\sqrt{\frac{55}{7}}\,\pi$		
				$K=3$					
(1023)	$\frac{64}{3}\sqrt{\frac{7}{5}}\,\pi$	(0132)	$-16\sqrt{10}\,\pi$	(1122)	$-\frac{64}{3}\sqrt{\frac{7}{5}}\,\pi$	(2112)	$-\frac{32}{5}\sqrt{21}\,\pi$	(2132)	$\frac{48}{5}\,\pi$
(0133)	$-16\sqrt{14}\,\pi$	(1123)	$-\frac{64}{3}\sqrt{\frac{14}{5}}\,\pi$	(2133)	$-12\sqrt{\frac{7}{5}}\,\pi$	(0134)	$-48\sqrt{2}\,\pi$	(2134)	$\frac{12}{\sqrt{5}}\,\pi$
(1221)	$\frac{64}{5}\sqrt{7}\,\pi$	(2211)	$32\sqrt{\frac{7}{15}}\,\pi$	(3221)	$-\frac{16}{5}\sqrt{2}\,\pi$	(1222)	$\frac{64}{3}\sqrt{\frac{14}{5}}\,\pi$	(2212)	$32\sqrt{\frac{14}{15}}\,\pi$
(2232)	$24\sqrt{\frac{2}{5}}\,\pi$	(3222)	$\frac{16}{\sqrt{5}}\,\pi$	(1223)	$\frac{64}{5}\sqrt{\frac{14}{3}}\,\pi$	(2233)	$4\sqrt{\frac{21}{5}}\,\pi$	(3223)	$-\frac{32}{5\sqrt{3}}\,\pi$
(2234)	$-4\sqrt{15}\,\pi$	(2310)	$-32\sqrt{\frac{7}{15}}\,\pi$	(2311)	$-32\sqrt{\frac{14}{15}}\,\pi$	(3321)	$-4\sqrt{\frac{14}{5}}\,\pi$	(2312)	$-\frac{32}{5}\sqrt{14}\,\pi$
(2332)	$\frac{16}{5}\sqrt{6}\,\pi$	(3322)	$-4\sqrt{\frac{14}{15}}\,\pi$	(2333)	$8\sqrt{\frac{21}{5}}\,\pi$	(3323)	$8\sqrt{\frac{14}{15}}\,\pi$	(2334)	$8\sqrt{\frac{33}{5}}\,\pi$
(3421)	$-4\sqrt{\frac{2}{5}}\,\pi$	(3422)	$-4\sqrt{\frac{10}{3}}\,\pi$	(3423)	$-8\sqrt{\frac{22}{15}}\,\pi$				
				$K=4$					
(1034)	$-24\sqrt{\frac{10}{7}}\,\pi$	(1133)	$36\sqrt{\frac{5}{7}}\,\pi$	(2123)	$144\sqrt{\frac{2}{35}}\,\pi$	(1134)	$60\sqrt{\frac{3}{7}}\,\pi$	(1232)	$-\frac{120}{7}\sqrt{6}\,\pi$
(2222)	$-32\sqrt{\frac{6}{7}}\,\pi$	(3212)	$-\frac{240}{7}\,\pi$	(3232)	$\frac{80}{7}\sqrt{\frac{3}{7}}\,\pi$	(1233)	$-60\sqrt{\frac{3}{7}}\,\pi$	(2223)	$-16\sqrt{\frac{30}{7}}\,\pi$
(3233)	$-\frac{40}{7}\sqrt{6}\,\pi$	(1234)	$-\frac{60}{7}\sqrt{11}\,\pi$	(3234)	$\frac{360}{7}\sqrt{\frac{2}{77}}\,\pi$	(2321)	$144\sqrt{\frac{2}{35}}\,\pi$	(3311)	$12\sqrt{\frac{30}{7}}\,\pi$
(2322)	$16\sqrt{\frac{30}{7}}\,\pi$	(3312)	$60\sqrt{\frac{2}{7}}\,\pi$	(3332)	$\frac{40}{7}\sqrt{6}\,\pi$	(2323)	$16\sqrt{\frac{66}{35}}\,\pi$	(3333)	$8\sqrt{\frac{15}{77}}\,\pi$
(3334)	$-\frac{360}{7\sqrt{11}}\,\pi$	(3410)	$-16\sqrt{\frac{15}{7}}\,\pi$	(3411)	$-60\sqrt{\frac{2}{7}}\,\pi$	(3412)	$-\frac{20}{7}\sqrt{66}\,\pi$	(3432)	$\frac{360}{7}\sqrt{\frac{2}{77}}\,\pi$
(3433)	$\frac{360}{7\sqrt{11}}\,\pi$	(3434)	$\frac{72}{7}\sqrt{\frac{195}{77}}\,\pi$						
				$K=5$					
(2134)	$-48\sqrt{\frac{22}{35}}\,\pi$	(2233)	$16\sqrt{\frac{33}{7}}\,\pi$	(3223)	$\frac{64}{7}\sqrt{\frac{55}{3}}\,\pi$	(2234)	$16\sqrt{\frac{33}{7}}\,\pi$	(2332)	$-\frac{16}{7}\sqrt{330}\,\pi$
(3322)	$-32\sqrt{\frac{22}{21}}\,\pi$	(2333)	$-16\sqrt{\frac{33}{7}}\,\pi$	(3323)	$-32\sqrt{\frac{22}{21}}\,\pi$	(2334)	$-\frac{16}{7}\sqrt{\frac{429}{5}}\,\pi$	(3421)	$64\sqrt{\frac{11}{35}}\,\pi$
(3422)	$32\sqrt{\frac{22}{21}}\,\pi$	(3423)	$\frac{32}{7}\sqrt{\frac{286}{15}}\,\pi$						
				$K=6$					
(3234)	$-80\sqrt{\frac{65}{231}}\,\pi$	(3333)	$20\sqrt{\frac{39}{11}}\,\pi$	(3334)	$20\sqrt{\frac{91}{33}}\,\pi$	(3432)	$-80\sqrt{\frac{65}{231}}\,\pi$	(3433)	$-20\sqrt{\frac{91}{33}}\,\pi$
(3434)	$-20\sqrt{\frac{65}{77}}\,\pi$								

Table 8. $\widetilde{\mathcal{C}}_{TT}^{00,\,K}(L'j'Lj)$ for $L_{\max}=3$.

$(L'j'Lj)$	$\widetilde{\mathcal{C}}_{TT}$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_{TT}$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_{TT}$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_{TT}$	$(L'j'Lj)$	$\widetilde{\mathcal{C}}_{TT}$
				$K=2$					
(1111)	$-4\sqrt{5}\,\pi$	(2121)	$\frac{12\,\pi}{\sqrt{5}}$	(1210)	$8\sqrt{\frac{5}{3}}\,\pi$	(3210)	$\frac{8\sqrt{\frac{5}{7}}\,\pi}{3}$	(1211)	$4\sqrt{15}\,\pi$
(2221)	$4\sqrt{3}\,\pi$	(3211)	$-\frac{8\sqrt{\frac{5}{7}}\,\pi}{3}$	(1212)	$4\sqrt{\frac{35}{3}}\,\pi$	(2222)	$4\sqrt{\frac{15}{7}}\,\pi$	(3212)	$\frac{8\sqrt{5}\,\pi}{21}$
(3232)	$\frac{32\sqrt{\frac{15}{7}}\,\pi}{7}$	(2321)	$8\sqrt{\frac{3}{35}}\,\pi$	(3311)	$\frac{8\sqrt{\frac{10}{7}}\,\pi}{3}$	(2322)	$8\sqrt{\frac{6}{7}}\,\pi$	(3312)	$-\frac{8\sqrt{\frac{5}{7}}\,\pi}{3}$
(3332)	$\frac{8\sqrt{15}\,\pi}{7}$	(2323)	$48\sqrt{\frac{2}{35}}\,\pi$	(3333)	$6\sqrt{\frac{10}{7}}\,\pi$	(3412)	$\frac{8\sqrt{15}\,\pi}{7}$	(3432)	$\frac{8\sqrt{\frac{5}{7}}\,\pi}{7}$
(3433)	$\frac{10\sqrt{10}\,\pi}{7}$	(3434)	$\frac{10\sqrt{\frac{330}{7}}\,\pi}{7}$						
				$K=3$					
(1221)	$-8\sqrt{\frac{14}{5}}\,\pi$	(2211)	$-\frac{8\sqrt{14}\,\pi}{3}$	(3221)	$4\sqrt{\frac{6}{5}}\,\pi$	(2212)	$-\frac{16\sqrt{7}\,\pi}{3}$	(3222)	$-4\sqrt{3}\,\pi$
(2310)	$\frac{8\sqrt{14}\,\pi}{3}$	(2311)	$\frac{16\sqrt{7}\,\pi}{3}$	(3321)	$\sqrt{42}\,\pi$	(2312)	$16\sqrt{\frac{7}{15}}\,\pi$	(2332)	$-\frac{8\,\pi}{\sqrt{5}}$
(3322)	$\sqrt{14}\,\pi$	(3323)	$-2\sqrt{14}\,\pi$	(3421)	$\sqrt{6}\,\pi$	(3422)	$5\sqrt{2}\,\pi$	(3423)	$2\sqrt{22}\,\pi$
				$K=4$					
(2222)	$\frac{32\,\pi}{\sqrt{7}}$	(3212)	$\frac{60\sqrt{3}\,\pi}{7}$	(3232)	$-\frac{80\,\pi}{7\sqrt{7}}$	(2321)	$-48\sqrt{\frac{3}{35}}\,\pi$	(3311)	$-9\sqrt{\frac{10}{7}}\,\pi$
(2322)	$-16\sqrt{\frac{5}{7}}\,\pi$	(3312)	$-15\sqrt{\frac{6}{7}}\,\pi$	(3332)	$-\frac{40\sqrt{2}\,\pi}{7}$	(2323)	$-16\sqrt{\frac{11}{35}}\,\pi$	(3333)	$-8\sqrt{\frac{5}{77}}\,\pi$
(3410)	$12\sqrt{\frac{5}{7}}\,\pi$	(3411)	$15\sqrt{\frac{6}{7}}\,\pi$	(3412)	$\frac{15\sqrt{22}\,\pi}{7}$	(3432)	$-\frac{120\sqrt{\frac{6}{77}}\,\pi}{7}$	(3433)	$-\frac{120\sqrt{\frac{3}{11}}\,\pi}{7}$
(3434)	$-\frac{72\sqrt{\frac{65}{77}}\,\pi}{7}$								
				$K=5$					
(2332)	$4\sqrt{\frac{110}{7}}\,\pi$	(3322)	$4\sqrt{11}\,\pi$	(3323)	$4\sqrt{11}\,\pi$	(3421)	$-4\sqrt{\frac{66}{5}}\,\pi$	(3422)	$-4\sqrt{11}\,\pi$
(3423)	$-4\sqrt{\frac{143}{35}}\,\pi$								
				$K=6$					
(3333)	$-6\sqrt{\frac{65}{11}}\,\pi$	(3432)	$40\sqrt{\frac{13}{77}}\,\pi$	(3433)	$2\sqrt{\frac{455}{11}}\,\pi$	(3434)	$10\sqrt{\frac{39}{77}}\,\pi$		

Limiting the multipolarity to $L_{\max}=3$ and thus $0\le K\le 6$ because $K\le 2\,L_{\max}$, the nonvanishing coefficients for $K>0$ are listed in tables 5 to 8. One should note that $\widetilde{\mathcal{C}}_{L}^{00,\,K}(L'j'Lj)=0$ for $L'+L+K=$odd, and $\widetilde{\mathcal{C}}_{T}^{00,\,K}(L'j'Lj)=0$ for $L'=L$ and $K=$odd. In view of the symmetry relations in (39) we list for $\widetilde{\mathcal{C}}_{L}^{00,\,K}(L'j'Lj)$, $\widetilde{\mathcal{C}}_{T}^{00,\,K}(L'j'Lj)$ and $\widetilde{\mathcal{C}}_{TT}^{00,\,K}(L'j'Lj)$ only the values for $j\le j'$ and for $j=j'$ only the ones for $L\le L'$. The other can be obtained from (39).

For the differential cross-section ($(\alpha'\alpha)=(00)$), the coefficients $\widetilde{\mathcal{D}}$ become quite simple. From (48) one gets

$$\widetilde{\mathcal{D}}_{00}^{K0}(\mu'j'\mu j)=\frac{(-)^{j'}}{2}\Bigg(\delta_{\mu',\mu}\,\delta_{\mu,2}\,(-)^{j'+j+K}\begin{pmatrix} j' & j & K\\ 0 & 0 & 0\end{pmatrix} -\sum_{l'l}\hat{l}'\hat{l}\begin{pmatrix} l' & l & K\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} j' & j & K\\ l & l' & 1\end{Bmatrix}U^{j'}_{l'1,\mu'}\,U^{j}_{l1,\mu}\Bigg), \qquad \text{(C.3)}$$

and in particular for $K=0$

$$\widetilde{\mathcal{D}}_{00}^{00}(\mu'j'\mu j)=\delta_{\mu',\mu}\,\delta_{j',j}\,\frac{1}{2\,\hat{j}}\,. \qquad \text{(C.4)}$$

The remaining nonvanishing coefficients $\widetilde{\mathcal{D}}$ are listed in tables 9 to 16 for $j\le j'\le 4$ and for $1\le K\le 8$, because for a given maximal multipolarity $L_{\max}$ the maximum j-value is $j_{\max}=L_{\max}+1$ and $K\le 2\,j_{\max}$. In these tables, we have already made use of the selection rules contained in the $3j$-symbols in (C.3). This means that a coefficient vanishes if $j'+j+K=$odd for $\mu'=\mu=2,4$ and $\mu',\mu\in\{1,3\}$, and furthermore if $j'+j+K=$even for $\mu'=1,3$ and $\mu=4$ and vice versa. For $j=j'$ only the coefficients for $\mu\le\mu'$ are listed because of (49). Again the coefficients with $j>j'$ follow from the listed ones using (49).

These tables allow one to determine explicitly the contributions of the various multipole moments to the coefficients of the expansion of the structure functions in terms of the $d^K_{m'm}(\theta)$-functions.

Table 9. $\widetilde{\mathcal{D}}^{10}_{00}(\mu' j' \mu j)$ for $j_{\max}=4$.

$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$	$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$
(1130)	$\frac{-\cos\epsilon_1+\sqrt{2}\sin\epsilon_1}{6}$	(2120)	$\frac{1}{2\sqrt{3}}$
(3130)	$\frac{\sqrt{2}\cos\epsilon_1+\sin\epsilon_1}{6}$	(1211)	$\frac{10\cos\epsilon_1\cos\epsilon_2+\sin\epsilon_1(-\sqrt{2}\cos\epsilon_2+6\sqrt{3}\sin\epsilon_2)}{60}$
(1231)	$\frac{-10\cos\epsilon_2\sin\epsilon_1+\cos\epsilon_1(-\sqrt{2}\cos\epsilon_2+6\sqrt{3}\sin\epsilon_2)}{60}$	(2221)	$\frac{1}{\sqrt{30}}$
(3211)	$\frac{6\sqrt{3}\cos\epsilon_2\sin\epsilon_1+(-10\cos\epsilon_1+\sqrt{2}\sin\epsilon_1)\sin\epsilon_2}{60}$	(3231)	$\frac{10\sin\epsilon_1\sin\epsilon_2+\cos\epsilon_1(6\sqrt{3}\cos\epsilon_2+\sqrt{2}\sin\epsilon_2)}{60}$
(4241)	$\frac{1}{2\sqrt{10}}$	(1312)	$\frac{21\sqrt{2}\cos\epsilon_2\cos\epsilon_3+\sin\epsilon_2(-\sqrt{3}\cos\epsilon_3+30\sin\epsilon_3)}{210}$
(1332)	$\frac{-21\sqrt{2}\cos\epsilon_3\sin\epsilon_2+\cos\epsilon_2(-\sqrt{3}\cos\epsilon_3+30\sin\epsilon_3)}{210}$	(2322)	$\frac{1}{2}\sqrt{\frac{3}{35}}$
(3312)	$\frac{30\cos\epsilon_3\sin\epsilon_2+(-21\sqrt{2}\cos\epsilon_2+\sqrt{3}\sin\epsilon_2)\sin\epsilon_3}{210}$	(3332)	$\frac{21\sqrt{2}\sin\epsilon_2\sin\epsilon_3+\cos\epsilon_2(30\cos\epsilon_3+\sqrt{3}\sin\epsilon_3)}{210}$
(4342)	$\sqrt{\frac{2}{105}}$	(1413)	$\frac{18\sqrt{3}\cos\epsilon_3\cos\epsilon_4+\sin\epsilon_3(-\cos\epsilon_4+14\sqrt{5}\sin\epsilon_4)}{252}$
(1433)	$\frac{-18\sqrt{3}\cos\epsilon_4\sin\epsilon_3+\cos\epsilon_3(-\cos\epsilon_4+14\sqrt{5}\sin\epsilon_4)}{252}$	(2423)	$\frac{1}{3\sqrt{7}}$
(3413)	$\frac{14\sqrt{5}\cos\epsilon_4\sin\epsilon_3+(-18\sqrt{3}\cos\epsilon_3+\sin\epsilon_3)\sin\epsilon_4}{252}$	(3433)	$\frac{18\sqrt{3}\sin\epsilon_3\sin\epsilon_4+\cos\epsilon_3(14\sqrt{5}\cos\epsilon_4+\sin\epsilon_4)}{252}$
(4443)	$\frac{1}{4}\sqrt{\frac{5}{21}}$		

Table 10. $\widetilde{\mathcal{D}}^{20}_{00}(\mu' j' \mu j)$ for $j_{\max}=4$.

$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$	$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$
(1111)	$-\frac{\sin\epsilon_1(-4\cos\epsilon_1+\sqrt{2}\sin\epsilon_1)}{4\sqrt{15}}$	(2121)	$-\frac{1}{\sqrt{30}}$
(3111)	$-\frac{-4\cos 2\epsilon_1+\sqrt{2}\sin 2\epsilon_1}{8\sqrt{15}}$	(3131)	$-\frac{\cos\epsilon_1(\sqrt{2}\cos\epsilon_1+4\sin\epsilon_1)}{4\sqrt{15}}$
(4141)	$\frac{1}{2\sqrt{30}}$	(1230)	$\frac{-(\sqrt{2}\cos\epsilon_2)+\sqrt{3}\sin\epsilon_2}{10}$
(1212)	$\frac{-15\sqrt{2}+\sqrt{2}\cos 2\epsilon_2+4\sqrt{3}\sin 2\epsilon_2}{40\sqrt{35}}$	(2220)	$\frac{1}{2\sqrt{5}}$
(2222)	$-\frac{1}{\sqrt{70}}$	(3230)	$\frac{\sqrt{3}\cos\epsilon_2+\sqrt{2}\sin\epsilon_2}{10}$
(3212)	$\frac{4\sqrt{105}\cos 2\epsilon_2-\sqrt{70}\sin 2\epsilon_2}{1400}$	(3232)	$-\frac{15\sqrt{2}+\sqrt{2}\cos 2\epsilon_2+4\sqrt{3}\sin 2\epsilon_2}{40\sqrt{35}}$
(4242)	$-\frac{1}{2\sqrt{70}}$	(1311)	$\frac{7\sqrt{15}\cos\epsilon_1\cos\epsilon_3+\sqrt{10}\sin\epsilon_1(-(\sqrt{3}\cos\epsilon_3)+9\sin\epsilon_3)}{210}$
(1331)	$\frac{-7\sqrt{15}\cos\epsilon_3\sin\epsilon_1-\sqrt{10}\cos\epsilon_1(\sqrt{3}\cos\epsilon_3-9\sin\epsilon_3)}{210}$	(1313)	$\frac{-49\sqrt{3}+\sqrt{3}\cos 2\epsilon_3+12\sin 2\epsilon_3}{168\sqrt{35}}$
(2321)	$\frac{1}{2}\sqrt{\frac{3}{35}}$	(2323)	$-\frac{1}{\sqrt{105}}$
(3311)	$\frac{9\sqrt{10}\cos\epsilon_3\sin\epsilon_1+\sqrt{15}(-7\cos\epsilon_1+\sqrt{2}\sin\epsilon_1)\sin\epsilon_3}{210}$	(3331)	$\frac{7\sqrt{15}\sin\epsilon_1\sin\epsilon_3+\sqrt{10}\cos\epsilon_1(9\cos\epsilon_3+\sqrt{3}\sin\epsilon_3)}{210}$
(3313)	$-\frac{-12\cos 2\epsilon_3+\sqrt{3}\sin 2\epsilon_3}{168\sqrt{35}}$	(3333)	$-\frac{49\sqrt{3}+\sqrt{3}\cos 2\epsilon_3+12\sin 2\epsilon_3}{168\sqrt{35}}$
(4341)	$\frac{1}{\sqrt{70}}$	(4343)	$-\frac{1}{4}\sqrt{\frac{3}{35}}$
(1412)	$\frac{27\cos\epsilon_2\cos\epsilon_4+\sqrt{6}\sin\epsilon_2(-\cos\epsilon_4+5\sqrt{5}\sin\epsilon_4)}{90\sqrt{7}}$	(1432)	$-\frac{27\cos\epsilon_4\sin\epsilon_2+\sqrt{6}\cos\epsilon_2(\cos\epsilon_4-5\sqrt{5}\sin\epsilon_4)}{90\sqrt{7}}$
(1414)	$\frac{-111\sqrt{5}+\sqrt{5}\cos 2\epsilon_4+20\sin 2\epsilon_4}{360\sqrt{77}}$	(2422)	$\frac{1}{\sqrt{70}}$
(2424)	$-\frac{1}{3}\sqrt{\frac{5}{77}}$	(3412)	$\frac{5\sqrt{30}\cos\epsilon_4\sin\epsilon_2+(-27\cos\epsilon_2+\sqrt{6}\sin\epsilon_2)\sin\epsilon_4}{90\sqrt{7}}$
(3432)	$\frac{27\sin\epsilon_2\sin\epsilon_4+\sqrt{6}\cos\epsilon_2(5\sqrt{5}\cos\epsilon_4+\sin\epsilon_4)}{90\sqrt{7}}$	(3414)	$-\frac{-20\cos 2\epsilon_4+\sqrt{5}\sin 2\epsilon_4}{360\sqrt{77}}$
(3434)	$-\frac{111\sqrt{5}+\sqrt{5}\cos 2\epsilon_4+20\sin 2\epsilon_4}{360\sqrt{77}}$	(4442)	$\frac{1}{2\sqrt{21}}$
(4444)	$-\frac{17}{12\sqrt{385}}$		

Table 11. $\widetilde{\mathcal{D}}^{30}_{00}(\mu' j' \mu j)$ for $j_{\max} = 4$.

$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$	$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$
(1211)	$\frac{9\cos\epsilon_2\sin\epsilon_1+\sqrt{3}\,(5\cos\epsilon_1-2\sqrt{2}\sin\epsilon_1)\sin\epsilon_2}{30\sqrt{7}}$	(1231)	$\frac{-5\sqrt{3}\sin\epsilon_1\sin\epsilon_2+\cos\epsilon_1\,(9\cos\epsilon_2-2\sqrt{6}\sin\epsilon_2)}{30\sqrt{7}}$
(2221)	$-\frac{1}{2}\sqrt{\frac{3}{35}}$	(3211)	$\frac{5\sqrt{3}\cos\epsilon_1\cos\epsilon_2-\sin\epsilon_1\,(2\sqrt{6}\cos\epsilon_2+9\sin\epsilon_2)}{30\sqrt{7}}$
(3231)	$-\frac{5\sqrt{3}\cos\epsilon_2\sin\epsilon_1+\cos\epsilon_1\,(2\sqrt{6}\cos\epsilon_2+9\sin\epsilon_2)}{30\sqrt{7}}$	(4241)	$\frac{1}{2\sqrt{35}}$
(1330)	$\frac{-\sqrt{3}\cos\epsilon_3+2\sin\epsilon_3}{14}$	(1312)	$\frac{2\sin\epsilon_2\,(4\sqrt{3}\cos\epsilon_3-15\sin\epsilon_3)+\sqrt{2}\cos\epsilon_2\,(-18\cos\epsilon_3+5\sqrt{3}\sin\epsilon_3)}{420}$
(1332)	$\frac{\cos\epsilon_2\,(8\sqrt{3}\cos\epsilon_3-30\sin\epsilon_3)+\sqrt{2}\sin\epsilon_2\,(18\cos\epsilon_3-5\sqrt{3}\sin\epsilon_3)}{420}$	(2320)	$\frac{1}{2\sqrt{7}}$
(2322)	$-\frac{1}{\sqrt{105}}$	(3330)	$\frac{2\cos\epsilon_3+\sqrt{3}\sin\epsilon_3}{14}$
(3312)	$\frac{\sqrt{2}\cos\epsilon_2\,(5\sqrt{3}\cos\epsilon_3+18\sin\epsilon_3)-2\sin\epsilon_2\,(15\cos\epsilon_3+4\sqrt{3}\sin\epsilon_3)}{420}$	(3332)	$-\frac{\sqrt{2}\sin\epsilon_2\,(5\sqrt{3}\cos\epsilon_3+18\sin\epsilon_3)+2\cos\epsilon_2\,(15\cos\epsilon_3+4\sqrt{3}\sin\epsilon_3)}{420}$
(4342)	$-\frac{1}{2\sqrt{210}}$	(1411)	$\frac{6\cos\epsilon_1\cos\epsilon_4+\sqrt{2}\sin\epsilon_1\,(-\cos\epsilon_4+2\sqrt{5}\sin\epsilon_4)}{12\sqrt{21}}$
(1431)	$-\frac{6\cos\epsilon_4\sin\epsilon_1+\sqrt{2}\cos\epsilon_1\,(\cos\epsilon_4-2\sqrt{5}\sin\epsilon_4)}{12\sqrt{21}}$	(1413)	$\frac{9\sin\epsilon_3\,(\cos\epsilon_4-2\sqrt{5}\sin\epsilon_4)+2\sqrt{3}\cos\epsilon_3\,(-11\cos\epsilon_4+\sqrt{5}\sin\epsilon_4)}{126\sqrt{22}}$
(1433)	$\frac{9\cos\epsilon_3\,(\cos\epsilon_4-2\sqrt{5}\sin\epsilon_4)-2\sqrt{3}\sin\epsilon_3\,(-11\cos\epsilon_4+\sqrt{5}\sin\epsilon_4)}{126\sqrt{22}}$	(2421)	$\frac{1}{3\sqrt{7}}$
(2423)	$-\frac{1}{\sqrt{154}}$	(3411)	$\frac{2\sqrt{10}\cos\epsilon_4\sin\epsilon_1+(-6\cos\epsilon_1+\sqrt{2}\sin\epsilon_1)\sin\epsilon_4}{12\sqrt{21}}$
(3431)	$\frac{6\sin\epsilon_1\sin\epsilon_4+\sqrt{2}\cos\epsilon_1\,(2\sqrt{5}\cos\epsilon_4+\sin\epsilon_4)}{12\sqrt{21}}$	(3413)	$\frac{-9\sin\epsilon_3\,(2\sqrt{5}\cos\epsilon_4+\sin\epsilon_4)+2\sqrt{3}\cos\epsilon_3\,(\sqrt{5}\cos\epsilon_4+11\sin\epsilon_4)}{126\sqrt{22}}$
(3433)	$-\frac{9\cos\epsilon_3\,(2\sqrt{5}\cos\epsilon_4+\sin\epsilon_4)+2\sqrt{3}\sin\epsilon_3\,(\sqrt{5}\cos\epsilon_4+11\sin\epsilon_4)}{126\sqrt{22}}$	(4441)	$\frac{1}{6}\sqrt{\frac{5}{14}}$
(4443)	$-\frac{1}{2}\sqrt{\frac{5}{462}}$		

Table 12. $\widetilde{\mathcal{D}}^{40}_{00}(\mu' j' \mu j)$ for $j_{\max} = 4$.

$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$	$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$
(1212)	$\frac{{\sin\epsilon_2}^2}{3\sqrt{70}}-\frac{\sin 2\epsilon_2}{\sqrt{105}}$	(2222)	$\frac{1}{\sqrt{70}}$
(3212)	$-\frac{\cos 2\epsilon_2}{\sqrt{105}}+\frac{\sin 2\epsilon_2}{6\sqrt{70}}$	(3232)	$\frac{{\cos\epsilon_2}^2}{3\sqrt{70}}+\frac{\sin 2\epsilon_2}{\sqrt{105}}$
(4242)	$-\frac{1}{3}\sqrt{\frac{2}{35}}$	(1311)	$\frac{18\sqrt{2}\cos\epsilon_3\sin\epsilon_1+\sqrt{3}\,(14\cos\epsilon_1-5\sqrt{2}\sin\epsilon_1)\sin\epsilon_3}{252}$
(1331)	$\frac{-14\sqrt{3}\sin\epsilon_1\sin\epsilon_3+\cos\epsilon_1\,(18\sqrt{2}\cos\epsilon_3-5\sqrt{6}\sin\epsilon_3)}{252}$	(1313)	$-\frac{-49+5\cos 2\epsilon_3+20\sqrt{3}\sin 2\epsilon_3}{84\sqrt{154}}$
(2321)	$-\frac{1}{3\sqrt{7}}$	(2323)	$\frac{1}{\sqrt{154}}$
(3311)	$\frac{14\sqrt{3}\cos\epsilon_1\cos\epsilon_3-\sqrt{2}\sin\epsilon_1\,(5\sqrt{3}\cos\epsilon_3+18\sin\epsilon_3)}{252}$	(3331)	$\frac{-14\sqrt{3}\cos\epsilon_3\sin\epsilon_1-\sqrt{2}\cos\epsilon_1\,(5\sqrt{3}\cos\epsilon_3+18\sin\epsilon_3)}{252}$
(3313)	$\frac{5\,(-4\sqrt{3}\cos 2\epsilon_3+\sin 2\epsilon_3)}{84\sqrt{154}}$	(3333)	$\frac{49+5\cos 2\epsilon_3+20\sqrt{3}\sin 2\epsilon_3}{84\sqrt{154}}$
(4341)	$\frac{1}{2\sqrt{42}}$	(4343)	$\frac{1}{6\sqrt{154}}$
(1430)	$\frac{-2\cos\epsilon_4+\sqrt{5}\sin\epsilon_4}{18}$		
(1412)	$\frac{6\sqrt{231}\sin\epsilon_2\,(5\cos\epsilon_4-4\sqrt{5}\sin\epsilon_4)+\sqrt{154}\cos\epsilon_2\,(-55\cos\epsilon_4+14\sqrt{5}\sin\epsilon_4)}{13860}$		
(1432)	$\frac{\sqrt{2}\sin\epsilon_2\,(55\cos\epsilon_4-14\sqrt{5}\sin\epsilon_4)+6\sqrt{3}\cos\epsilon_2\,(5\cos\epsilon_4-4\sqrt{5}\sin\epsilon_4)}{180\sqrt{77}}$	(1414)	$-\frac{-27+\cos 2\epsilon_4+4\sqrt{5}\sin 2\epsilon_4}{12\sqrt{2002}}$
(2420)	$\frac{1}{6}$	(2422)	$-\frac{1}{3}\sqrt{\frac{5}{77}}$
(2424)	$\frac{3}{\sqrt{2002}}$	(3430)	$\frac{\sqrt{5}\cos\epsilon_4+2\sin\epsilon_4}{18}$
(3412)	$\frac{-6\sqrt{3}\sin\epsilon_2\,(4\sqrt{5}\cos\epsilon_4+5\sin\epsilon_4)+\sqrt{2}\cos\epsilon_2\,(14\sqrt{5}\cos\epsilon_4+55\sin\epsilon_4)}{180\sqrt{77}}$		
(3432)	$-\frac{6\sqrt{3}\cos\epsilon_2\,(4\sqrt{5}\cos\epsilon_4+5\sin\epsilon_4)+\sqrt{2}\sin\epsilon_2\,(14\sqrt{5}\cos\epsilon_4+55\sin\epsilon_4)}{180\sqrt{77}}$		
(3414)	$\frac{-4\sqrt{5}\cos 2\epsilon_4+\sin 2\epsilon_4}{12\sqrt{2002}}$	(3434)	$\frac{27+\cos 2\epsilon_4+4\sqrt{5}\sin 2\epsilon_4}{12\sqrt{2002}}$
(4442)	$-\frac{1}{2\sqrt{462}}$	(4444)	$\frac{3}{2\sqrt{2002}}$

Table 13. $\widetilde{\mathcal{D}}^{50}_{00}(\mu' j' \mu j)$ for $j_{\max}=4$.

$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$	$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$
(1312)	$\frac{-5\sqrt{66}\cos\epsilon_3\sin\epsilon_2-7\sqrt{33}\cos\epsilon_2\sin\epsilon_3+3\sqrt{22}\sin\epsilon_2\sin\epsilon_3}{462}$	(1332)	$\frac{7\sqrt{33}\sin\epsilon_2\sin\epsilon_3+\cos\epsilon_2(-5\sqrt{66}\cos\epsilon_3+3\sqrt{22}\sin\epsilon_3)}{462}$
(2322)	$\sqrt{\frac{5}{462}}$	(3312)	$\frac{-7\sqrt{33}\cos\epsilon_2\cos\epsilon_3+\sqrt{22}\sin\epsilon_2(3\cos\epsilon_3+5\sqrt{3}\sin\epsilon_3)}{462}$
(3332)	$\frac{7\sqrt{33}\cos\epsilon_3\sin\epsilon_2+\sqrt{22}\cos\epsilon_2(3\cos\epsilon_3+5\sqrt{3}\sin\epsilon_3)}{462}$	(4342)	$-\frac{1}{2}\sqrt{\frac{5}{231}}$
(1411)	$\frac{\sqrt{10}\cos\epsilon_4\sin\epsilon_1+(3\cos\epsilon_1-\sqrt{2}\sin\epsilon_1)\sin\epsilon_4}{6\sqrt{33}}$	(1431)	$\frac{\sqrt{2}\cos\epsilon_1(\sqrt{5}\cos\epsilon_4-\sin\epsilon_4)-3\sin\epsilon_1\sin\epsilon_4}{6\sqrt{33}}$
(1413)	$\frac{2\sqrt{3}\cos\epsilon_3(13\sqrt{5}\cos\epsilon_4-28\sin\epsilon_4)+9\sin\epsilon_3(-5\sqrt{5}\cos\epsilon_4+14\sin\epsilon_4)}{252\sqrt{143}}$		
(1433)	$\frac{-9\cos\epsilon_3(5\sqrt{5}\cos\epsilon_4-14\sin\epsilon_4)+2\sqrt{3}\sin\epsilon_3(-13\sqrt{5}\cos\epsilon_4+28\sin\epsilon_4)}{252\sqrt{143}}$		
(2421)	$-\frac{1}{6}\sqrt{\frac{5}{11}}$	(2423)	$\sqrt{\frac{5}{1001}}$
(3411)	$\frac{3\cos\epsilon_1\cos\epsilon_4-\sqrt{2}\sin\epsilon_1(\cos\epsilon_4+\sqrt{5}\sin\epsilon_4)}{6\sqrt{33}}$	(3431)	$-\frac{3\cos\epsilon_4\sin\epsilon_1+\sqrt{2}\cos\epsilon_1(\cos\epsilon_4+\sqrt{5}\sin\epsilon_4)}{6\sqrt{33}}$
(3413)	$\frac{9\sin\epsilon_3(14\cos\epsilon_4+5\sqrt{5}\sin\epsilon_4)-2\sqrt{3}\cos\epsilon_3(28\cos\epsilon_4+13\sqrt{5}\sin\epsilon_4)}{252\sqrt{143}}$		
(3433)	$\frac{9\cos\epsilon_3(14\cos\epsilon_4+5\sqrt{5}\sin\epsilon_4)+2\sqrt{3}\sin\epsilon_3(28\cos\epsilon_4+13\sqrt{5}\sin\epsilon_4)}{252\sqrt{143}}$		
(4441)	$\frac{1}{3\sqrt{22}}$	(4443)	$\frac{1}{4\sqrt{3003}}$

Table 14. $\widetilde{\mathcal{D}}^{60}_{00}(\mu' j' \mu j)$ for $j_{\max}=4$.

$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$	$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$
(1313)	$-\frac{5\sin\epsilon_3(-12\cos\epsilon_3+\sqrt{3}\sin\epsilon_3)}{12\sqrt{1001}}$	(2323)	$-\frac{5}{\sqrt{3003}}$
(3313)	$-\frac{5(-12\cos 2\epsilon_3+\sqrt{3}\sin 2\epsilon_3)}{24\sqrt{1001}}$	(3333)	$\frac{-5\cos\epsilon_3(\sqrt{3}\cos\epsilon_3+12\sin\epsilon_3)}{12\sqrt{1001}}$
(4343)	$\frac{5}{4}\sqrt{\frac{3}{1001}}$	(1412)	$-\frac{5\cos\epsilon_4\sin\epsilon_2}{3\sqrt{429}}-\frac{3\cos\epsilon_2\sin\epsilon_4}{\sqrt{1430}}+\frac{7\sin\epsilon_2\sin\epsilon_4}{6\sqrt{2145}}$
(1432)	$\frac{3\sin\epsilon_2\sin\epsilon_4}{\sqrt{1430}}+\frac{\cos\epsilon_2(-50\cos\epsilon_4+7\sqrt{5}\sin\epsilon_4)}{30\sqrt{429}}$	(1414)	$\frac{-57\sqrt{5}+7\sqrt{5}\cos 2\epsilon_4+140\sin 2\epsilon_4}{360\sqrt{143}}$
(2422)	$\frac{1}{2}\sqrt{\frac{5}{143}}$	(2424)	$-\frac{1}{3}\sqrt{\frac{5}{143}}$
(3412)	$-\frac{3\cos\epsilon_2\cos\epsilon_4}{\sqrt{1430}}+\frac{\sin\epsilon_2(7\sqrt{5}\cos\epsilon_4+50\sin\epsilon_4)}{30\sqrt{429}}$	(3432)	$\frac{3\cos\epsilon_4\sin\epsilon_2}{\sqrt{1430}}+\frac{\cos\epsilon_2(7\sqrt{5}\cos\epsilon_4+50\sin\epsilon_4)}{30\sqrt{429}}$
(3414)	$-\frac{7(-20\cos 2\epsilon_4+\sqrt{5}\sin 2\epsilon_4)}{360\sqrt{143}}$	(3434)	$-\frac{57\sqrt{5}+7\sqrt{5}\cos 2\epsilon_4+140\sin 2\epsilon_4}{360\sqrt{143}}$
(4442)	$-\sqrt{\frac{2}{429}}$	(4444)	$\frac{1}{12\sqrt{715}}$

Table 15. $\widetilde{\mathcal{D}}^{70}_{00}(\mu' j' \mu j)$ for $j_{\max}=4$.

$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$	$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$
(1413)	$\frac{7\sqrt{5}\cos\epsilon_4\sin\epsilon_3+(9\sqrt{3}\cos\epsilon_3-4\sin\epsilon_3)\sin\epsilon_4}{18\sqrt{143}}$	(1433)	$\frac{\cos\epsilon_3(7\sqrt{5}\cos\epsilon_4-4\sin\epsilon_4)-9\sqrt{3}\sin\epsilon_3\sin\epsilon_4}{18\sqrt{143}}$
(2423)	$-\frac{1}{6}\sqrt{\frac{35}{143}}$	(3413)	$\frac{9\sqrt{3}\cos\epsilon_3\cos\epsilon_4-\sin\epsilon_3(4\cos\epsilon_4+7\sqrt{5}\sin\epsilon_4)}{18\sqrt{143}}$
(3433)	$-\frac{9\sqrt{3}\cos\epsilon_4\sin\epsilon_3+\cos\epsilon_3(4\cos\epsilon_4+7\sqrt{5}\sin\epsilon_4)}{18\sqrt{143}}$	(4443)	$\frac{1}{2}\sqrt{\frac{7}{429}}$

Table 16. $\widetilde{\mathcal{D}}^{80}_{00}(\mu' j' \mu j)$ for $j_{\max}=4$.

$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$	$(\mu' j' \mu j)$	$\widetilde{\mathcal{D}}$
(1414)	$\frac{7\sin\epsilon_4(-20\cos\epsilon_4+\sqrt{5}\sin\epsilon_4)}{15\sqrt{4862}}$	(2424)	$\frac{7}{3}\sqrt{\frac{5}{4862}}$
(3414)	$\frac{7(-20\cos 2\epsilon_4+\sqrt{5}\sin 2\epsilon_4)}{30\sqrt{4862}}$	(3434)	$\frac{7\cos\epsilon_4(\sqrt{5}\cos\epsilon_4+20\sin\epsilon_4)}{15\sqrt{4862}}$
(4444)	$-\frac{14}{3}\sqrt{\frac{2}{12155}}$		